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Quadratic Funding under Incomplete Information

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Abstract

Quadratic funding is a recently proposed mechanism for the provision of continuous public public goods, that has been shown to provide socially optimal outcomes under complete information. In this work, I analyze this mechanism in a more general informational context, and contribute to the existing literature in three main ways. First, I develop a more rigorous framework for the quadratic funding mechanism under complete information, and prove both existence results and more general efficiency results. Second, I relax the assumption of complete information, and evaluate both the existence of equilibria and efficiency of the mechanism in this new setting. I show that, contrary to what was conjectured by its proposers, the quadratic funding mechanism is generally not efficient under incomplete information. Third, I propose and use two numeric measures to calculate the size of this mechanism's inefficiency in a variety of situations. Among my findings, some of the main factors that are associated with a larger inefficiency include a larger population size, higher valuation of the public good by individuals, and larger variance in the expected value of contributions to the mechanism.

Keywords: game theory, public good provision, incomplete information, efficiency, quadratic funding mechanism

JEL codes: C72, D82, H41

Resumo

O *quadratic funding* é uma proposta recente de mecanismo para a provisão de bens públicos contínuos, e que se mostrou capaz de gerar o nível socialmente ótimo de provisão de bens públicos sob informação completa. Neste trabalho, eu me dedico à análise desse mecanismo em um contexto informacional mais geral, e contribuo para a literatura existente de três maneiras principais. Primeiro, eu desenvolvo um arcabouço mais rigoroso para o *quadratic funding* sob informação completa, e demonstro tanto resultados de existência quanto resultados mais gerais de eficiência. Em segundo lugar, eu relaxo a suposição de informação completa, e avalio a existência e a eficiência dos equilíbrios do mecanismo nesse novo contexto. Eu provo que, ao contrário do que foi conjecturado pelos proponentes do mecanismo, em geral o *quadratic funding* não é eficiente sob informação incompleta. Em terceiro lugar, eu desenvolvo e utilizo duas medidas numéricas para calcular o tamanho da ineficiência do mecanismo em várias situações. Eu encontro, dentre outros resultados, que níveis altos de ineficiência estão associados a situações em que há um grande número de jogadores, em que indivíduos têm alta valoração pelo bem público, e que o valor esperado das contribuições individuais possui variância elevada.

Palavras-chave: teoria dos jogos, provisão de bens públicos, informação incompleta, eficiência, *quadratic funding*

Códigos JEL: C72, D82, H41

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1. Introduction

Finding a solution to the problem of public good provision is a key topic within the field of economics. By definition, public goods are non-rival and non-excludable, so when a group of individuals provide a public good, everyone can benefit from it. A classic result in public sector economics shows that when continuous public goods are provided privately, the quantity provided is usually below the socially optimal level. Thus, a Pareto improvement can be achieved if individuals are able to use a mechanism that enables them to reach the socially optimal level of funding for public goods.

One plausible solution to this problem would be to use what is called a Lindahl tax, which works as follows. If a government knows its citizens' demand function for the public good, it could provide the level of the public good that equates marginal cost and the aggregate marginal willingness to pay, and tax each individual the quantity of the public good provided multiplied by her marginal willingness to pay for the public good at the provision level. Under a reasonably weak set of assumptions, this level of provision is socially optimal, and individuals would pay for the public good an amount equivalent to what they would pay for a private good, in a way that would make everyone better off than the situation where the public good is provided privately, and the taxes collected would suffice to pay for the good. However, even if we assume that individuals know their demands for public goods, there is no guarantee that individuals would truthfully reveal them to the government. In fact, even if all other individuals were to report truthfully, an individual would generally have an incentive to under-report his preferences for the public good, since in doing so he could still benefit from the public good at the level provided by others while still being able to spend the amount that would be spent with taxes on other goods. Therefore, the implementation of this solution would also lead to underprovision of the public good, as in the case of private provision.

Thus, designing a mechanism that is capable of generating a socially optimal level of funding for public goods is not a trivial task, and doing so in a way that is compatible with individual incentives was generally believed to be impossible in the field until a few decades ago (Groves and Ledyard 1977). Since then, several incentive compatible mechanisms that provide the optimal level of continuous public goods have been proposed, but all of them seem to lack some important property that would enable broad applicability.

An example of mechanism in this literature is the one presented by Groves and Ledyard (1977). This mechanism is the first in this literature, and besides reaching an efficient provision, it also has the important property of achieving a balanced budget. Nevertheless, it fails to be individually rational in the general case, meaning that some individuals might prefer to abstain from participation, a severe limitation for contexts where voluntary engagement is required. Walker (1981) proposes a variation of this mechanism that not only reaches efficiency, but also does not violate individual rationality. Nevertheless, the resulting mechanism is subject to the problem of unstable equilibria, an important constraint on its practical implementation (Healy 2006). Tideman and Plassmann (2017) make a detailed discussion of mechanisms in this literature, as well as their common characteristics.

Two other mechanisms related to this literature are worth mentioning. In the context of providing a *discrete* public good, the renowned Vickrey-Clarke-Groves mechanism makes revealing one's true valuation for the public good a dominant strategy, but is extremely susceptible to collusion (Ausubel and Milgrom 2005). On the other hand, in practical terms a frequently used mechanism is that of majority voting, where each person would choose a funding level, and provision is made based on the median choice. This mechanism, however, fails to deliver efficiency in general, since it is based on median rather than average preference (Bowen 1943).

More recently, Buterin, Hitzig, and Weyl (2019) proposed a mechanism, known as quadratic funding,[1] that works by providing a level of the public good that is equal to the square of the sum of the square roots of individual contributions, i.e., if every individual i in the population makes a contribution $c_i \geq 0$ to a public good, then the total funding for this good is $(\sum_i c_i^{1/2})^2$. The authors have defended that this mechanism, besides enabling the provision of an efficient level of public goods, also benefits from some properties that would make it suitable for application in contexts where other mechanisms do not perform well.

They argue that a fundamental property of the quadratic funding mechanism is that would not require any assumptions about both the set of public goods to be funded, and about the number of individuals that would contribute to these public goods. They claim that this property is not shared by other mechanisms in the literature, and would allow quadratic funding to be particularly well-suited for situations where one would wish for the individuals themselves to propose public goods for funding, rather than letting these individuals choose contributions for a fixed set of public goods. Some types of public good for which this property would be particularly relevant, according to the authors, include open source software, projects that benefit minority groups within cities and political campaigns.

Other desirable properties of the mechanism include that it is homogeneous of degree one, and so avoids incentives to split or merge projects for public goods for a desire of increased funding and avoids problems with inflation; that it is individually rational; and that it is equal to a private contributions scheme when only a single individual contributes to the public good. The mechanism also presents some important drawbacks, such as vulnerability to collusion and a budget deficit, that make it less suitable to implementation in some contexts.

From a theoretical perspective, there is one additional reason for interest in the quadratic funding mechanism. One can think of a public good provision mechanism as a production function for public goods, where individual contributions represent different inputs for the production process. The goal of a technology would be to aggregate individual preferences in such a way that the inputs chosen by individuals

[1] Even though Buterin, Hitzig, and Weyl (2019) refer to this mechanism as quadratic *finance*, it is more widely known as quadratic funding, so I use the latter term in this work.

would result in an optimal provision level. From this perspective, quadratic funding belongs to the class of funding mechanisms with a constant elasticity of substitution. The proposers of quadratic funding prove that it is the only mechanism in this class satisfying efficiency.

To demonstrate the efficiency of quadratic funding, Buterin, Hitzig, and Weyl (2019) assume a setting of complete information, i.e., the utility functions of agents would be common knowledge. The authors claim that efficiency does not rely heavily on this assumption, but they do not provide any evidence for this claim. Given that in practical applications of this mechanism an assumption of complete information is frequently not realistic, a more thorough understanding of the behavior of quadratic funding under incomplete information is essential for determining under which conditions this mechanism could provide an efficient outcome, and which adaptations might be necessary for this new informational setting. A deeper understanding of this mechanism under incomplete information is also important for comparing it with other mechanisms in this literature, since efficiency under incomplete information could give it an important advantage in terms of implementation on many different contexts.

This work seeks to evaluate the efficiency of quadratic funding under incomplete information. In the following chapter I present the mechanism in a complete information framework similar to the one used by Buterin, Hitzig, and Weyl (2019), and then present a more general proof of efficiency and some existence results. In the third chapter, I adapt the framework so as to allow for incomplete information and put forward existence and efficiency results, showing in particular that quadratic funding usually fails to be efficient under incomplete information. Motivated by this result, in the fourth chapter I develop measures of inefficiency and use two different game types to calculate, numerically, the effect of changing the values of various parameters on efficiency. I use the theoretical results and the economic intuition developed previously to analyze my findings, allowing me to draw several important conclusions with respect to contexts where quadratic funding is more or less appropriate. In the fifth chapter, I close by summing up my findings and presenting the most relevant theoretical and practical implications they have.

2. Complete Information

In this chapter, I present the quadratic funding mechanism (henceforth QF) under complete information and analyze some of its properties, which serve as a stepping stone for my further analysis under incomplete information. The first section introduces the framework that is used in this chapter and defines the QF mechanism. In the second section, I present some existence and efficiency proofs for QF and, in particular, generalize the analysis done by Buterin, Hitzig, and Weyl (2019), showing that efficiency applies more broadly than stated in their work. Lastly, in the third section, I present an analysis of QF when individuals have constant relative risk aversion (CRRA) utility functions for the consumption of the public good, which is relevant to my analyses using this class of functions under incomplete information.

2.1 Framework

This section follows sections 3 and 4 of Buterin, Hitzig, and Weyl (2019). I simplify the analysis presented there in two ways. First, their work develops the mechanism in a setting that enables it to be used where several public goods are to be funded simultaneously. However, the funding of each public good can be treated independently without any loss to the capacity of generalizing the analysis to the case of several public goods. Therefore, I present and evaluate the mechanism as if used to fund only a single public good.

Second, Buterin, Hitzig, and Weyl (2019) consider the possibility for an eventual budget deficit of the mechanism to be financed by some form of taxation. The reason for them doing so is that this is relevant for some extensions of the QF mechanism that they propose. As I do not analyze any of these extensions, I abstain from introducing the possibility of taxation to individuals as a way to finance budget deficits, and instead assume that the central planner already has enough resources to finance any budget deficit. This is a realistic assumption in some scenarios, such as when the central planner is a foundation with a large endowment, but an extension of the results presented here to the case where taxation is possible is analogous. Nevertheless, I note that QF has a budget deficit whenever two or more individuals make positive contributions, which is easy to verify using the mechanism's definition.

I assume that there are $n \geq 2$ individuals, for which I define the set $\mathcal{N} := \{1, \dots, n\}$, and a single public good to be provided. Each individual $i \in \mathcal{N}$ is characterized by a quasilinear utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined as $u_i(F, m) = v_i(F) + m$, where the linear good is the numeraire and $v_i : \mathbb{R}_+ \to \mathbb{R}$ represents the monetary-equivalent utility of consumption of the public good. I assume every function v_i is C^1, strictly increasing, and strictly concave. Moreover, in addition to the hypothesis adopted by Buterin, Hitzig, and Weyl (2019), I assume that for every $i \in \mathcal{N}$ we have that $\lim_{F\to\infty} v_i'(F) = 0$. The reason for doing so, as I show in proposition 2.8, is that this hypothesis guarantees the existence of an efficient provision level.[1] Below, I present the definition of a funding mechanism.

Definition 2.1. A *funding mechanism* is a function $\Phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ that, for any profile of individual contributions to the public good $\mathbf{c} := (c_1, c_2, \dots, c_n) \in \mathbb{R}^n_+$, determines a level of public good provision $\Phi(\mathbf{c}) = F \in \mathbb{R}_+$.

In words, a funding mechanism is simply a rule for deciding the level of public good provision based on contributions chosen by individuals. For example, one could simply take the sum of all contributions as the level of funding for the public good ($\Phi(\mathbf{c}) = \sum_{i=1}^n c_i$), which would constitute a private provision scheme for the public good. A funding mechanism can be thought of as a technology for the production

[1] The assumption that $\lim_{F\to\infty} v_i'(F) = 0$ for every $i \in \mathcal{N}$ can be weakened. For example, if we assume instead that for all i there exists $F \geq 0$ such that $v_i'(F) < 1/n$, an analogous demonstration to that of proposition 2.8 would apply.

of public goods, such that the contribution of each individual represents a different input, and the mechanism aggregates the inputs to transform them into the funding level output. The goal of the central planner is to use a production function that incentivizes individuals to choose inputs in a way that generates an efficient level of output, which will be defined later on in this section. Before delving into efficiency, I define the concept of a game, and that of a Nash equilibrium for the game.

Definition 2.2. A *game* of public good provision with a funding mechanism Φ under complete information is defined by

$$\mathcal{G} = \{(v_i)_{i\in\mathcal{N}}, \Phi\}.$$

Definition 2.3. An allocation $\mathbf{c}^* \in \mathbb{R}^n_+$ is an *equilibrium for* $\mathcal{G}$ if, for all $i \in \mathcal{N}$, we have that $v_i(\Phi(c_i^*, \mathbf{c}_{-i}^*)) - c_i^* \geq v_i(\Phi(z, \mathbf{c}_{-i}^*)) - z$, for all $z \geq 0$. Alternatively, $c_i^* = \arg\max_{z\geq 0} v_i(\Phi(z, \mathbf{c}_{-i}^*)) - z$.

Thus, an equilibrium for this game is a vector of contributions such that no individual can unilaterally be better off by choosing a contribution that is different from the equilibrium one. I call an equilibrium *interior* if all contributions are strictly positive, and a *corner equilibrium* otherwise.

Now I turn to the definition of efficiency, or optimality. The social welfare function in this context is

$$W(F) = \left(\sum_{i=1}^{n} v_i(F)\right) - F.$$

In words, the total social welfare is the difference between the aggregate monetary-equivalent utility of consuming the public good and the cost of providing said funding level. This definition captures the intuition that funding for the public good comes at the expense of societal resources, in this case those of the of individuals or the central planner.

Naturally, a desirable property for a funding mechanism is the ability to maximize the social welfare function. I formalize this notion with the following definitions.

Definition 2.4. A funding level $F^e \geq 0$ is said to be a *socially optimal provision*, or *efficient provision*, if

$$F^e = \arg\max_{F\geq 0} \left(\sum_{i=1}^{n} v_i(F)\right) - F.$$

That is, it maximizes total social welfare. The first order conditions imply that $\sum_{i=1}^{n} v_i'(F^e) \leq 1$, with equality holding when $F^e > 0$.

Definition 2.5. For the game $\{(v_i)_{i\in\mathcal{N}}, \Phi\}$, the funding mechanism Φ is *optimal*, or *efficient*, if there exists some efficient provision F^e, and some equilibrium contribution profile $\mathbf{c}^*$, such that $\Phi(\mathbf{c}^*) = F^e$.

The latter definition is my own, and differs from that presented by Buterin, Hitzig, and Weyl (2019). Their definition of optimality is akin to definition 2.4, and is presented as if there was only a single possible equilibrium level of public good provision for funding mechanisms. As I show in the next section, this assumption turns out not to be true even for QF. Therefore, definition 2.5 is more appropriate for mechanism optimality.

I now define QF, which is the funding mechanism of interest to my analysis.

Definition 2.6. The *quadratic funding mechanism* is specified by the function $\Phi^{QF} : \mathbb{R}_+^n \to \mathbb{R}_+$ defined as

$$\Phi^{QF}(\mathbf{c}) = \left(\sum_{i=1}^{n} (c_i)^{1/2}\right)^2 .$$

Note that this mechanism provides the same level of funding as a private provision scheme if only a single individual makes a strictly positive contribution, which is expected to happen for a good that only benefits a single individual in the population. Additionally, note that if n individuals contribute the same amount, the mechanism provides n^2 times the individual contribution amounts, indicating that the mechanism provides a matching for contributions that is proportional to the number of contributing individuals.

Following my earlier interpretation of funding mechanisms as a production function, it is relevant to note that QF belongs to the class of mechanisms with a constant elasticity of substitution (CES) between inputs.

Definition 2.7. A *CES funding mechanism with a substitution parameter* $\rho \in (0, 1]$ is specified by the function $\Phi^\rho : \mathbb{R}_+^n \to \mathbb{R}_+$ defined as

$$\Phi^\rho(\mathbf{c}) = \left(\sum_{i=1}^{n} (c_i)^\rho\right)^{1/\rho} .$$

This class of mechanisms works exactly as a CES production function, and $1/\rho$ indicates the extent to which aggregate provision grows with the number of contributing individuals. QF represents the particular case where $\rho = 1/2$ which, as I have illustrated before, represents the case where n equal contributions generate a provision of n^2 times the individual contributions. A private provision scheme is also a particular case of this class of mechanisms, and is represented by $\rho = 1$, reflecting the fact that when n individuals make an equal contribution, the provision level for the public good is n^1 times the individual contributions. CES mechanisms with $\rho < 1/2$ then provide a larger matching than QF for increases in the contributing population, while those with $\rho > 1/2$ would have a lower matching.

2.2 Existence and Efficiency

In this section, I present some existence and efficiency results for the quadratic funding mechanism, which serve as a support for the analysis I present under incomplete information. I begin by showing that an optimal allocation exists and, in fact, is unique.

Proposition 2.8. *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1, strictly concave, and $\lim_{F\to\infty} v_i'(F) = 0$. Then, there exists a unique efficient provision $F^e \geq 0$.*

See proof on page 48.

I now evaluate the efficiency of QF. First, I introduce the optimality result presented by Buterin, Hitzig, and Weyl (2019), adapted to the definitions introduced in the previous section.

Proposition 2.9 (Buterin, Hitzig, and Weyl 2019, adapted)**.** *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1, strictly increasing and strictly concave. Then, interior equilibria for the quadratic funding mechanism provide the socially optimal level of the public good.*

Proof. The problem that an individual $i \in \mathcal{N}$ faces is given by

$$\max_{c_i \geq 0} v_i \left(\left[\sum_{j=1}^{n} c_j^{1/2} \right]^2 \right) - c_i,$$

whose first order conditions for interior solutions are

$$\frac{2v_i'(\Phi^{QF}(\mathbf{c})) \cdot \left[\sum_{j=1}^{n} c_j^{1/2} \right]}{2c_i^{1/2}} = 1.$$

Rearranging , we get

$$v_i'(\Phi^{QF}(\mathbf{c})) = \frac{c_i^{1/2}}{\sum_{j=1}^{n} c_j^{1/2}}. \tag{2.1}$$

Adding equation (2.1) for all i yields $\sum_{i=1}^{n} v_i'(\Phi^{QF}(\mathbf{c})) = 1$, as desired. □

However, there are two ways in which this result is not sufficient to guarantee efficiency in the sense introduced by definition 2.5. First, it does not consider the case where $F^e = 0$, in which case the contribution of every individual must be zero, and therefore a corner solution, for the mechanism to be optimal. Second, it does not provide sufficient conditions for the existence of efficient equilibria, as it only shows that the existence of interior equilibria implies efficiency. The following result complements proposition 2.9 by demonstrating efficiency as defined in the previous section.

Proposition 2.10. *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1, strictly increasing, strictly concave, and $\lim_{F \to \infty} v_i'(F) = 0$. Then, the quadratic funding mechanism is optimal.*

See proof on page 48.

In particular, since definition 2.5 requires existence, this result also guarantees the existence of equilibria for QF under the hypotheses adopted.

Given the previous result, it is natural to ask whether QF is efficient in a stronger sense than that presented in proposition 2.10, namely, if every equilibrium of QF is efficient. In the general case, given the hypotheses introduced in the previous section, the answer turns out to be negative, as I show in proposition 2.12. Before proving this result, however, I present an auxiliary lemma.

Lemma 2.11. *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1 and strictly increasing. Then, the only possible corner equilibrium for the quadratic funding mechanism is the allocation $\mathbf{0} \in \mathbb{R}^n_+$.*

See proof on page 49.

Proposition 2.12. *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1, strictly increasing, strictly concave, and $\lim_{F\to\infty} v_i'(F) = 0$. Then, the quadratic fundig mechanism has inefficient equilibria if, and only if, zero is not the efficient provision and $v_i'(0) \leq 1$ for all $i \in \mathcal{N}$.*

See proof on page 50.

Remark. When $F^e > 0$, the interior equilibrium presented in the proof of proposition 2.10 is the unique interior equilibrium. Thus, using lemma 2.11, we can see that under the hypotheses adopted here there exists at least one and at most two equilibria for QF. When there is a single equilibrium, it generates the efficient provision. When there are two equilibria, as in the conditions specified by proposition 2.12, one of them is the null vector.

I conclude this section by presenting the result that QF is the only efficient mechanism in the class of CES funding mechanisms.

Proposition 2.13 (Buterin, Hitzig, and Weyl 2019, adapted)**.** *Suppose that, for all $i \in \mathcal{N}$, we have that v_i is C^1, strictly increasing and strictly concave. Then, a CES funding mechanism has efficient interior equilibria if, any only if, $\rho = 2$.*

2.3 CRRA Utility Functions

In this section, I present an interpretation for different parameter values of the constant relative risk aversion (CRRA) utility function in the specific context where it is used as the utility function for the public good in a setting where the QF mechanism is used. A CRRA utility function is a function $v : \mathbb{R}_+ \to \mathbb{R}$ defined by $v(F) = \beta F^{1-\gamma}/(1-\gamma)$, where $\beta > 0$ and $\gamma > 0$ are parameters, and $v(F) = \beta \ln(F)$ when $\gamma = 1$. As I show below, this class of functions has an important property when used in a context where QF is used, justifying my use of CRRA functions in subsequent chapters.

The optimization problem that the individual indexed by 1 faces when QF is used is given by

$$\max_{c_1 \geq 0} v_1(\Phi^{QF}(\mathbf{c})) - c_1,$$

whose first order conditions for an interior solution can be written as

$$\frac{dc_1}{dc_1} = v_1'(\Phi^{QF}(\mathbf{c})) \cdot \frac{\partial \Phi^{QF}(\mathbf{c})}{\partial c_1}. \tag{2.2}$$

That is, the (unitary) marginal cost of contribution, in the left-hand side, must be equal to the marginal utility of consumption for the public good times the marginal increase in provision for a marginal increase in contribution. Equation (2.2) can be rewritten as

$$\left[\frac{\partial \Phi^{QF}(\mathbf{c})}{\partial c_1}\right]^{-1} = v_1'(\Phi^{QF}(\mathbf{c})). \tag{2.3}$$

Letting v_1 be a CRRA utility function and using the definition of Φ^{QF}, we get

$$\frac{c_1^{1/2}}{\sum_{j=1}^{n} c_j^{1/2}} = \frac{\beta}{\left(\sum_{j=1}^{n} c_j^{1/2}\right)^{2\gamma}}. \tag{2.4}$$

Let $\bar{c}_1$ be individual 1's best response to some contribution vector for other individuals $\bar{\mathbf{c}}_{-1}$, such that $(\bar{c}_1, \bar{\mathbf{c}}_{-1})$ satisfies equation (2.4). If we change the contributions of the other individuals to a vector $\mathbf{c}_{-1}$ in such a way that $\sum_{j=2}^{n} c_j^{1/2} > \sum_{j=2}^{n} \bar{c}_j^{1/2}$, it results from equation (2.4) that $\left[\partial \Phi^{QF}(\bar{c}_1, \mathbf{c}_{-1})/\partial c_1\right]^{-1} \geq v_1'(\Phi^{QF}(\bar{c}_1, \mathbf{c}_{-1}))$ if, and only if, $\gamma \geq 1/2$. Since the left-hand side of equation (2.4) is strictly increasing in c_1, and the right-hand side is strictly decreasing in c_1, the individual's best response to $\mathbf{c}_{-1}$ is $c_1 \leq \bar{c}_1$ if, and only if, $\gamma \geq 1/2$.

In the economic terms used to interpret equation (2.2), the above analysis can be summarized as follows. When $\gamma > 1/2$, an increase in the aggregate contribution of the other individuals causes the marginal utility of consumption of the public good to fall proportionally more than the increase in the marginal change in provision of the public good for a change in contribution. Therefore, the marginal utility of the original contribution diminishes, and so the individual's best response is then to lower her original contribution. On the other hand, when $\gamma < 1/2$, an increase in the aggregate contribution of the other individuals causes the marginal utility of consumption of the public good to fall proportionally *less* than the increase in the marginal provision of the public good for a given contribution. Hence, the marginal utility of the original contribution increases, and to maintain the equality between marginal cost and marginal benefit the individual must increase her contribution. Lastly, when $\gamma = 1/2$, an increase in the aggregate contribution of the other individuals does not affect the marginal utility of the individual's original contribution, and thus the optimal contribution stays the same. In other words, when $\gamma = 1/2$, the optimal individual contribution is *independent* of the contribution of the other individuals.

I summarize the above discussion with the following proposition.

Proposition 2.14. *Suppose that an individual has a CRRA utility function for the public good. Then, the individual's contribution to the public good is a decreasing function of the sum of the square root of the contributions of other individuals under complete information if, and only if,* $\gamma \geq 1/2$.

See proof on page 50.

Remark. In particular, proposition 2.14 and the assumptions adopted in section 2.1 imply that, if an individual has a CRRA utility function for the public good with $\gamma < 1/2$, then her contribution to the public good is always higher than her consumption of this good if it were a private good instead (i.e., if she were to play this game alone instead). The opposite happens when $\gamma > 1/2$, and in the intermediate case where $\gamma = 1/2$, her contribution to the public good is always equal to what would be her private contribution.

3. Incomplete Information

I now turn to the evaluation of QF under incomplete information. In the first section, I introduce the hypotheses I use for incomplete information, and subsequently adjust the definitions used under complete information to this context. In section 3.2, I demonstrate some existence and efficiency results for this adaptation, showing that under incomplete information QF is only efficient in very specific situations, and contrasting these results with the case of complete information.

3.1 Framework

As before, I assume that there are $n \geq 2$ individuals, and define the set $\mathcal{N} := \{1, \ldots, n\}$. Each individual $i \in \mathcal{N}$ is characterized by a finite set of types $\Theta_i = \{\theta_i^\ell ; 1 \leq \ell \leq L_i, L_i \in \mathbb{N}\}$, and an expected utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \times \Theta_i \to \mathbb{R}$ defined as $u_i(F, m; \theta_i) = v_i(F; \theta_i) + m$, where the linear good is the numeraire and $v_i : \mathbb{R}_+ \times \Theta_i \to \mathbb{R}$ represents the monetary-equivalent expected utility of the individual for a level $F \geq 0$ of funding for the public good when this individual's type is $\theta_i \in \Theta_i$. I use $\Theta = \times_{i=1}^n \Theta_i$ as the set of all possible states of the world. The function $\Pr : \Theta \to [0, 1]$ is a probability distribution that defines the probability of any given state of the world $\theta \in \Theta$ being realized, and is assumed to be known by all individuals. In an analogous way to the case of complete information, I assume that, for every $i \in \mathcal{N}$ and every $\theta_i \in \Theta_i$, we have that $v_i(\cdot; \theta_i)$ is C^1, strictly increasing, strictly concave, and $\lim_{F \to \infty} v_i'(F; \theta_i) = 0$.

Note that the introduction of incomplete information does not pose any problem to the notion of funding mechanism introduced in definition 2.1. Therefore, I use the same definition of funding mechanism in the present framework. The introduction of incomplete information does, however, require adjusting most of the other definitions previously used. I start by redefining a game, adding the individual types and the probability distribution function to definition 2.2.

Definition 3.1. A *game* of public good provision with a funding mechanism Φ under incomplete information is defined by

$$\mathcal{G} = \left\{ (v_i, \Theta_i)_{i \in \mathcal{N}}, \Pr, \Phi \right\}.$$

Naturally, the definition of a Nash equilibrium for a game of complete information also requires adjustment so as to become a Bayes-Nash equilibrium for a game of incomplete information, as I do below.

Definition 3.2. Let $\mathbf{c} : \Theta \to \mathbb{R}^n_+$ be some function such that, for every state of the world $\theta \in \Theta$, determines a strategy profile $\mathbf{c}(\theta) = (c_1(\theta_1), \ldots, c_n(\theta_n)) \in \mathbb{R}^n_+$. We say that $\mathbf{c}^*$ is an *equilibrium for* $\mathcal{G}$ if, for all $i \in \mathcal{N}$, all $\theta_i \in \Theta_i$ and all $z \in \mathbb{R}_+$ we have

$$E\left[v_i(\Phi(c_i^*(\theta_i), \mathbf{c}^*_{-i}(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - c_i^*(\theta) \geq E\left[v_i(\Phi(z, \mathbf{c}^*_{-i}(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - z.$$

Alternatively, $\mathbf{c}^*$ is such that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$,

$$c_i^*(\theta_i) = \arg\max_{z \in \mathbb{R}_+} E\left[v_i(\Phi(z, \mathbf{c}^*_{-i}(\theta_{-i})); \theta_i) \,\middle|\, \theta_i \right] - z.$$

I now adjust redefine efficiency under incomplete information. The welfare function, in this case, is a function $W : \mathbb{R}_+ \times \Theta \to \mathbb{R}$ defined by

$$W(F; \theta) = \left(\sum_{i=1}^n v_i(F; \theta_i) \right) - F.$$

With the same underlying logic of the welfare function introduced in section 2.1. In this scenario, however, there is not necessarily a single funding level $F \geq 0$ that maximizes welfare in all states of the world, since there is no guarantee that the $F \geq 0$ that maximizes W on one state of the world would also do so in the others. Thus, the efficient funding level can vary with the state of the world. In this sense, the relevant concept of an efficient provision for a game is, in fact, dependent on the state of the world. In other words, it is a function that maps every $\theta \in \Theta$ to a funding level that maximizes $W(\cdot;\theta)$. Consequently, with regards to the funding mechanism, the relevant notion of efficiency is one where the optimal provision results as an equilibrium of the mechanism for every possible state of the world. I formally state these two definitions below.

Definition 3.3. We say that $F^e : \Theta \to \mathbb{R}_+$ is an *(ex-post) efficient provision for* $\mathcal{G}$ if, for all $\theta \in \Theta$,

$$F^e(\theta) = \arg\max_{F \geq 0} \left(\sum_{i=1}^{n} v_i\left(F; \theta_i\right) \right) - F.$$

That is, it maximizes the total social welfare for any given state of the world $\theta \in \Theta$. It follows from the first order conditions that, for all $\theta \in \Theta$, we have $\sum_{i=1}^{n} v_i'(F^e(\theta); \theta_i) \leq 1$, holding with equality when $F^e(\theta) > 0$.

Definition 3.4. For the game $\left\{(v_i, \Theta_i)_{i \in \mathcal{N}}, \Pr, \Phi\right\}$, the funding mechanism Φ is *optimal*, or *efficient*, if there exist an efficient provision function F^e and an equilibrium contribution profile $\mathbf{c}^*$ such that, for all $\theta \in \Theta$, we have $\Phi(\mathbf{c}^*(\theta)) = F^e(\theta)$.

This definition of efficiency is a direct adaptation of definition 2.5 when incorporating the new definition of efficient provision. Economically, a mechanism is efficient if it has an equilibrium that provides an efficient level of the public good for all possible states of the world.

Finally, note that QF in this framework can be defined exactly as in definition 2.6. For the sake of simplicity, I use $F(\theta) := \Phi^{QF}(\mathbf{c}(\theta))$ to denote the funding provided by QF when players choose some strategy profile $\mathbf{c}(\theta)$. If $\mathbf{c}^*$ is an equilibrium for QF, I use the notation $F^*(\theta) := \Phi^{QF}(\mathbf{c}^*(\theta))$.

The best response of individual $i \in \mathcal{N}$ with type $\theta_i \in \Theta_i$ who faces the contributions of all other individuals $\mathbf{c}_{-i}(\theta_i)$ is given by

$$c_i(\theta_i) = \arg\max_{z \geq 0} E\left[v_i\left(\Phi^{QF}(z, \mathbf{c}_{-i}(\theta_{-i})); \theta_i\right) \,\middle|\, \theta_i\right] - z. \tag{3.1}$$

Thus, if $\mathbf{c}^*$ is an interior equilibrium ($\mathbf{c}^*(\theta) \gg 0$ for all $\theta \in \Theta$), the first order conditions imply that it must satisfy, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$,

$$(c_i^*(\theta_i))^{1/2} = E\left[v_i'(F^*(\theta)); \theta_i) \cdot (F^*(\theta))^{1/2} \,\middle|\, \theta_i\right], \tag{3.2}$$

for all $i \in \mathcal{N}$. Multiplying equation (3.2) by $\Pr(\theta_i)$ and adding across all $\theta_i \in \Theta_i$ yields

$$E\left[(c_i^*(\theta_i))^{1/2}\right] = E\left[v_i'(F^*(\theta));\theta_i) \cdot (F^*(\theta))^{1/2}\right]. \tag{3.3}$$

Now, adding equation (3.3) for all i,

$$E\left[(F^*(\theta))^{1/2}\right] = E\left[\sum_{i=1}^{n} v_i'(F^*(\theta));\theta_i) \cdot (F^*(\theta))^{1/2}\right], \tag{3.4}$$

which can be written as

$$E\left[\left[\left(\sum_{i=1}^{n} v_i'(F^*(\theta));\theta_i)\right) - 1\right] \cdot (F^*(\theta))^{1/2}\right] = 0. \tag{3.5}$$

Note that equation (3.5) does not rule out the possibility that QF is efficient. In fact, if there exist a strategy profile $\mathbf{c}$ and an efficient provision F^e such that, for all $\theta \in \Theta$ we have $F^*(\theta) = F^e(\theta)$, then $\sum_{i=1}^{n} v_i'(F^*(\theta));\theta_i) = 1$ by definition of efficient provision, so equation (3.5) is satisfied. However, this condition is also weaker than that of efficiency, since efficiency would require specifically that the first term within the expected value to be equal to zero. Note that this stands in contrast with the situation of complete information, as in that case there would be no expected value in equation (3.5), and since $F^*(\theta) > 0$ by the hypothesis of an interior equilibrium, efficiency follows immediately. In the next section, I investigate if equilibria that satisfy equation (3.5) would necessarily be efficient and show that, in the general case, efficiency does not follow.

3.2 Existence and Efficiency

I start this section by setting forth some existence results, with respect both to efficient funding levels and to QF equilibria.

Proposition 3.5. *Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i)$ is C^1, strictly concave, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Then, there exists a unique efficient provision $F^e : \Theta \to \mathbb{R}_+$.*

See proof on page 51.

In the cases where the hypotheses of this proposition are valid, $F^e(\theta)$ refers to the unique efficient provision for the state of the world $\theta \in \Theta$.

I now prove that the assumptions outlined in the previous section are sufficient for the existence of an equilibrium for QF. Before doing so, I first introduce two lemmas. The first lemma shows conditions for the best response correspondence to be at most single-valued, and the second presents a compact set for which the best response correspondence is well-defined when taking this set as its domain and codomain.

Lemma 3.6. *Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i)$ is C^1, strictly increasing and strictly concave. Then, for any $i \in \mathcal{N}$ and any $\mathbf{c}_{-i} : \Theta_{-i} \to \mathbb{R}_+^{n-1}$, the best response correspondence $c_i(\cdot;\mathbf{c}_{-i}) : \Theta_i \to \mathbb{R}_+$ is at most single valued.*

See proof on page 52.

Lemma 3.7. *Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i)$ is C^1, strictly concave and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Then, there exists $A > 0$ such that, for any $i \in \mathcal{N}$ and any $\theta_i \in \Theta_i$, if the contributions of all other individuals belong to the interval $[0, A]$ for any profile of types, then i's best response cannot lie outside of this interval.*

See proof on page 53.

Proposition 3.8. *Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, we have that $v_i(\cdot;\theta_i)$ is C^1, strictly increasing, strictly concave and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Then, there exists an equilibrium for the quadratic funding mechanism.*

See proof on page 53.

Therefore, an equilibrium for this mechanism always exists for given the hypotheses introduced in the previous section. I now proceed to analyze under which situations an *optimal* equilibrium exists, i.e., when QF is efficient. I begin this analysis by completely characterizing efficiency in situations where there exist states of the world where the public good should not be provided in proposition 3.9.

Proposition 3.9. *For all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, let the function $v_i(\cdot;\theta_i)$ be C^1, strictly increasing and strictly concave, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Suppose that there exists some $\theta' \in \Theta$ such that $F^e(\theta') = 0$, and that $\Pr(\theta) > 0$ for all $\theta \in \Theta$. Then, the quadratic funding mechanism is efficient if, and only if, $F^e(\theta) = 0$ for all $\theta \in \Theta$.*

See proof on page 54.

This proposition stands in contrast with the efficiency result shown for QF under complete information, as it already shows that there is a broad range of games where the mechanism is inefficient. The context where inefficiency occurs here is one where there is no direct analogy under complete information, since in the latter case there is only a single possible state of the world, and so either the optimal funding is zero or the optimal funding is positive. On the other hand, this proposition also shows that QF is efficient when the socially optimal funding is zero in all states of the world, showing that the hypothesis of incomplete information does not eliminate efficiency entirely.

I now turn to the analysis of cases where the optimal provision is always positive. In the following proposition, I evaluate the simplest case of incomplete information, namely when only a single individual has multiple types.

Proposition 3.10. *For all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, let $v_i(\cdot;\theta_i)$ be C^1, strictly increasing and strictly concave, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Suppose that there exists a single $j \in \mathcal{N}$ such that $|\Theta_j| > 1$, and suppose we have $F^e(\theta) > 0$ for all $\theta \in \Theta$. Then, the quadratic funding mechanism is efficient if, and only if, there exists $A \in \mathbb{R}$ such that $\sum_{i\in\mathcal{N}\setminus\{j\}} v_i'(F^e(\theta);\theta_i^1) = A(F^e(\theta))^{-1/2}$ for all $\theta \in \Theta$. Furthermore, we have that $A = \sum_{i\in\mathcal{N}\setminus\{j\}} (c_i^*(\theta_i^1))^{1/2}$.*

See proof on page 55.

This necessary and sufficient condition for efficiency when only a single individual has multiple types introduced in this proposition makes it possible to easily verify whether QF is efficient in a broad range of games of this type. First, QF is efficient in cases where the efficient provision level for all states of the world is the same ($F^e(\theta) = F^e(\theta')$ for all $\theta, \theta' \in \Theta$), as in that case $\sum_{i\in\mathcal{N}\setminus\{j\}} v_i'(F^e(\theta);\theta_i^1)$ would be constant for all $\theta \in \Theta$. A second possibility is that, if the utility functions for the public good of every individual with a single type is a CRRA with $\gamma = 1/2$, then the condition on proposition 3.10 is satisfied for all $F > 0$, and in particular to all $F^e(\theta)$, so QF is efficient. This requirement can be relaxed, and efficiency would hold even if each individual with a single type did not have a CRRA utility for the public good with $\gamma = 1/2$, but the sum of their utility functions for the public good were to be a CRRA with $\gamma = 1/2$, as illustrated by the example below.

Example 3.11. Consider the case with three individuals, where only the first individual has more than one type. Let $v_2(F;\theta_2^1) = F^{1/2} + (F+1)^{1/6}$ and $v_3(F;\theta_3^1) = F^{1/2} - (F+1)^{1/6}$, and let $v_1(\cdot;\theta_1)$ satisfy the hypotheses presented in section 3.1 for all $\theta_1 \in \Theta_1$. Both v_2 and v_3 are strictly increasing and strictly concave. Since $v_2'(F;\theta_2^1) + v_3'(F;\theta_3^1) = 4F^{-1/2}$, proposition 3.10 guarantees the efficiency of QF.

On the other hand, QF is not efficient if all individuals with one type have a CRRA utility function for the public good with $\gamma \neq 1/2$ and there are at least two states of the world with different efficient provision levels. In fact, if QF is efficient in a situation where only a single individual has multiple types, then the sum of the utility functions for the public good of the individuals with one type can be "represented" by a CRRA with $\gamma = 1/2$ at every efficient provision level, since when the condition in proposition 3.10 is satisfied we have that $\sum_{i \in \mathcal{N}\setminus\{j\}} v_i'(\cdot;\theta_i^1)$ has the same derivative as the function $2AF^{1/2}$ in all provision levels.

The following proposition deals with the remaining case where the optimal funding level is positive for all states of the world and where possibly several individuals have multiple types. It presents a necessary condition for efficiency that is similar to, but stronger than, the one in proposition 3.10, as it imposes a requirement about the marginal utilities of every individual at the socially optimal provision levels.

Proposition 3.12. *For all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$, let $v_i(\cdot;\theta_i)$ be C^1, strictly increasing and strictly concave, and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. Suppose we have $F^e(\theta) > 0$ for all $\theta \in \Theta$, and that for all $i \in \mathcal{N}$ there exists a function $A_i : \Theta_i \to R$ such that $v_i'(F^e(\theta);\theta_i) = A_i(\theta_i) \cdot (F^e(\theta))^{-1/2}$. Then, the quadratic funding mechanism is efficient.*

See proof on page 56.

Similarly to the case where only one individual could have multiple types, this proposition implies that QF is efficient whenever every individual has a CRRA utility function for the public good with $\gamma = 1/2$. It also applies to other cases, but a general rule is that the greater the number of efficient provision levels, the more "similar" to a CRRA with $\gamma = 1/2$ the individual utility functions for the public good have to be for this result to apply, i.e., the individual marginal utility functions for the public good would need to coincide with that of a CRRA with $\gamma = 1/2$ for more provision levels. The condition presented in this proposition is not necessary and, in fact, example 3.11 shows that in the case of a single individual with multiple types, the mechanism can be efficient even if none of the individuals with a single type satisfies this sufficiency condition. However, in situations with more than a single individual with multiple types and several different efficient provision levels for different states of the world, it appears to be the case that QF is inefficient in most situations where proposition 3.12 does not hold. The next proposition illustrates this idea for the case where all utility functions for the public good are CRRA with the same γ, showing that efficiency only holds when $\gamma = 1/2$.

Proposition 3.13. *Suppose that for every $i \in \mathcal{N}$ and every $\theta_i \in \Theta_i$, we have that $v_i(F;\theta_i) = \beta_i(\theta_i)F^{1-\gamma}/(1-\gamma)$, where $\beta_i(\theta_i) > 0$ and $\gamma > 0$. Additionally, suppose that for some $j \in \mathcal{N}$, there exists $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$, and that $\Pr(\theta) > 0$ for all $\theta \in \Theta$. Then, the quadratic funding mechanism is efficient if, and only if, $\gamma = 1/2$.*

See proof on page 57.

In economic terms, the efficiency results presented above can be interpreted in the following way. We can see from the proof of proposition 3.12 that QF is efficient whenever, for each individual and each type, there is a single optimal contribution for that individual in all possible states of the world conditional on his type. In other words, QF is efficient if an individual would not change his contribution if he were to learn which state of the world is the true one. In cases where the optimal funding can be zero with positive probability, if the optimal funding can also be greater than zero with positive probability, conditional on an individual's type, then this individual would prefer to make at least two different contributions for different states of the world (one which is zero and one which is strictly positive), and thus this condition does not hold. If there is a single efficient equilibrium for all states of the world, then every individual can use this unique equilibrium value to compute their optimal contributions, and thus there is an efficient equilibrium.

A special case of efficiency is when individuals have CRRA utility functions for the public good with $\gamma = 1/2$. As I have shown in section 2.3, when an individual has a CRRA with $\gamma = 1/2$ under complete information, his optimal contribution to the public good is independent of the contribution of the other individuals. Similarly, under incomplete information, the contribution that satisfies this individual's first order conditions only depends on his own utility function for the public good, regardless of what others are contributing in different states of the world. In a scenario where every individual has a CRRA with $\gamma = 1/2$, therefore, QF is efficient. On the other hand, when an individual has a CRRA with $\gamma > 1/2$ ($\gamma < 1/2$), the individual wishes to decrease (increase) his contributions for states of the world where the equilibrium provision level is higher, and so efficiency does not follow.

This intuition presents a sufficient condition, but not a necessary one, as shown in example 3.11. In that case, if there are two different possible efficient provision levels, individuals 2 and 3 would want to make different contributions for states of the world with different efficient provision levels. The fact that they are unable to do so, in principle, could lead to inefficiency. Yet, due to the form of their utility functions for the public good, the overcontribution of one individual in one state of the world compensates for the undercontribution of the other individual in the same state, and so the equilibrium provision level turns out to be efficient.

The games where QF is efficient are very restricted, since most functions satisfying the hypotheses introduced in the previous section lead the individuals to have different optimal contributions in different states of the world. Cases where individual contributions perfectly compensate for each other so as to result in efficiency only occur in exceptional situations, as in example 3.11. Thus, in the general case, when using the QF mechanism in a framework with incomplete information, such mechanism is not efficient, contrarily to what Buterin, Hitzig, and Weyl (2019) conjectured.

4. Deadweight Loss

So far, I have shown that QF's efficiency fails to generalize to a case of incomplete information. In this chapter, I turn to the numerical estimation and analysis of the inefficiency, or deadweight loss, resulting from using the quadratic funding mechanism under incomplete information. I investigate how inefficiency varies in response to changes in parameters of the utility functions for the public good, in the probability of individual types, and in population size. I also evaluate the relative inefficiency generated by QF when compared to a simple benchmark provision level, and use it to evaluate whether the former always performs better than the latter and, when it does, by what extent.

On the first section, I explain the methodology employed in this chapter and the economic reasoning behind it. The results are presented and analyzed on the second and third sections, separated by games with two players and games with a varying population, respectively.

4.1 Methodology

The numerical analysis I present here relies on the theoretical basis established in the previous chapter, and so I adopt all the same framework introduced in section 3.1. Furthermore, to make the numerical estimations, I assume that every individual has a CRRA utility function for the public good, so as to let me to apply both the results and the intuition developed so far for this class of functions. The inefficiency measure I use is as follows.

Definition 4.1. For a given game $\mathcal{G}$, the *absolute deadweight loss* for the contribution-based quadratic funding mechanism is given by

$$\Delta W^A := E\left[W(F^e(\theta)) - W(F^*(\theta))\right].$$

In words, it is the expected difference in welfare from providing the QF equilibrium level instead of the efficient level of the public good for every state of the world. Taking the expected value for estimating welfare losses is a common approach in the incomplete information literature, such as Rustichini, Satterthwaite, and Williams (1994) and Vives (2002). I call this measure *absolute* deadweight loss to contrast it with a relative measure of deadweight loss that I define below. Before defining this relative measure, however, I introduce the concept of an *ex-ante* efficient provision level.

Definition 4.2. We say that $F^{EA} \geq 0$ is an *ex-ante efficient provision for* $\mathcal{G}$ if

$$F^{EA} = \underset{F \geq 0}{\arg\max}\, E\left[\left(\sum_{i=1}^{N} V_i\left(F;\theta_i\right)\right) - F\right].$$

That is, if it maximizes the expected total social welfare before the individual types are known. From the first order conditions we have that $\sum_{i=1}^{n} E\left[v_i'(F^{EA};\theta_i)\right] \leq 1$, holding with equality when $F^{EA} > 0$.

Remark. Given the hypotheses presented in section 3.1, it can be shown, in an analogous manner to propositions 2.8 and 3.5, that there exists a unique ex-ante efficient provision.

Note that this definition is for an *ex-ante* efficient provision, standing in contrast with the ex-post efficient provision, which, as I previously argued, is the most adequate measure of efficient public good funding under incomplete information. The concept of ex-ante efficient provision refers to the optimal level of public good provision given that one only knows the distribution of states of the world, but not the type of any individual. Therefore, since the distribution of types is common knowledge, it is reasonable to assume that the central planner knows the distribution of types, and would hence be able to provide the ex-ante efficient provision level. Consequently, this provision level can serve as a lower bound for what the central planner could do without using any mechanism to acquire information. One would expect QF to perform at least as well, in expectation, as the ex-ante optimal funding level, since under QF the contributing individuals make use of their private information. My second measure of inefficiency, which I define below, seeks to capture this idea.

Definition 4.3. For a given game $\mathcal{G}$, the *relative deadweight loss* for the contribution-based quadratic funding mechanism is given by

$$\Delta W^R := \frac{E\left[W(F^e(\theta)) - W(F^*(\theta))\right]}{E\left[W(F^e(\theta)) - W(F^{EA})\right]}.$$

The relative deadweight loss, therefore, measures the ratio between the absolute deadweight loss generated by QF and the absolute deadweight loss generated by providing the ex-ante efficient level of the public good. When this measure is lower than one, it indicates that using QF is better, in expectation, than providing the ex-ante optimal level of the public good, and the contrary holds when this measure is greater than one.

Despite the intuitive appeal of this relative inefficiency measure, I have been unable to find similar measures in the literature. Therefore, to the best of my knowledge, this inefficiency measure is a contribution that the present work brings to the literature.

I use two different kinds of games to make deadweight loss estimations. The first of them is the simplest possible case for incomplete information, i.e., only two individuals, one of them with two possible types, and the other with a single possible type. The simplicity of this game allows me to evaluate how each parameter affects the deadweight in isolation. On the other hand, the second kind of game seeks to evaluate inefficiency when the population increases, which is a particularly common question in the literature (e.g. Rustichini, Satterthwaite, and Williams 1994; Vives 2002; Lalley and Weyl 2019), and how does the relative risk aversion parameter interact with population size. I henceforth refer to the first of these kinds as games of "two players", and to the second type as "varying population" games.

Formally, in the games of two players, I set $n = 2$, $|\Theta_1| = 2$, and $|\Theta_2| = 1$. I let $\alpha := \Pr(\theta_1 = \theta_1^1)$. The utility functions for the public good of these individuals, given their possible types, are

$$v_1(F;\theta_1^1) = \frac{F^{1-\gamma}}{1-\gamma},$$
$$v_1(F;\theta_1^2) = \beta_1\frac{F^{1-\gamma}}{1-\gamma},$$
$$v_2(F;\theta_2^1) = \beta_2\frac{F^{1-\gamma}}{1-\gamma},$$

where $\beta_1, \beta_2 \in [0.1, 50]$. I now solve the individual problems. For the first individual, when $\theta_1 = \theta_1^1$,

$$\max_{c_1(\theta_1^1)\geq 0} \frac{(F(\theta_1^1,\theta_2^1))^{1-\gamma}}{1-\gamma} - c_1(\theta_1^1)$$
$$\Rightarrow \left(c_1(\theta_1^1)\right)^{1/2} = \left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}. \tag{4.1}$$

Analogously, when $\theta_1 = \theta_1^2$, we get

$$\left(c_1(\theta_1^2)\right)^{1/2} = \beta_1\left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}. \tag{4.2}$$

And for individual 2, we have

$$\max_{c_2(\theta_2^1)\geq 0} \alpha\left(\beta_2\frac{(F(\theta_1^1,\theta_2^1))^{1-\gamma}}{1-\gamma}\right) + (1-\alpha)\left(\beta_2\frac{(F(\theta_1^2,\theta_2^1))^{1-\gamma}}{1-\gamma}\right) - c_2(\theta_2^1)$$
$$\begin{aligned}\Rightarrow \left(c_2(\theta_2^1)\right)^{1/2} &= \alpha\beta_2\left[\left(c_1(\theta_1^1)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma} \\ &\quad + (1-\alpha)\beta_2\left[\left(c_1(\theta_1^2)\right)^{1/2} + \left(c_2(\theta_2^1)\right)^{1/2}\right]^{1-2\gamma}.\end{aligned} \tag{4.3}$$

The ex-post efficient level of the public good must satisfy, when $\theta = (\theta_1^1, \theta_2^1)$,

$$\max_{F\geq 0}(1+\beta_2)\frac{F^{1-\gamma}}{1-\gamma} - F,$$

which implies that

$$F^e(\theta_1^1,\theta_2^1) = (1+\beta_2)^{1/\gamma}. \tag{4.4}$$

Analogously, when $\theta = (\theta_1^2, \theta_2^1)$, we get

$$F^e(\theta_1^2,\theta_2^1) = (\beta_1+\beta_2)^{1/\gamma}. \tag{4.5}$$

Lastly, the ex-ante optimal provision level for the public good solves

$$\max_{F\geq 0}\alpha\left[\frac{F^{1-\gamma}}{1-\gamma}\right] + (1-\alpha)\left[\beta_1\frac{F^{1-\gamma}}{1-\gamma}\right] + \beta_2\frac{F^{1-\gamma}}{1-\gamma} - F,$$

from which it results that

$$F^{EA} = (\alpha + (1-\alpha)\beta_1 + \beta_2)^{1/\gamma}. \tag{4.6}$$

Hence, for some given parameter values, I use equations (4.1) to (4.3) to compute the individual contributions to QF, equations (4.4) and (4.5) to compute the ex-post optimal provision levels, and equation (4.6) to compute the ex-ante optimal provision level.

In the varying population games, I make the assumption that all individuals may have one of two types with the same utility functions. Specifically, for any individual $i \in \mathcal{N}$, let $|\Theta_i| = 2$, let $\Pr(\theta_i = \theta_i^1) = \Pr(\theta_i = \theta_i^1 \,|\, \theta_j) = 1/2$ for all $i, j \in \mathcal{N}$ with $i \neq j$, and let their utilities of consuming the public good for each type be given by

$$v_i(F; \theta_i^1) = \frac{F^{1-\gamma}}{1-\gamma}$$
$$v_i(F; \theta_i^2) = 2\frac{F^{1-\gamma}}{1-\gamma}.$$

Given the above setup, we have that the probability of $0 \leq k \leq n$ individuals being of type 1 follows a Bernoulli distribution. As individuals are symmetric, their contributions to the public good are identical whenever they have the same type, so I use the notation $x^1 := c_i(\theta_i^1)$ and $x^2 := c_i(\theta_i^2)$ to refer to these contributions. The first order conditions for some individual with type 1 can be written as

$$(x^1)^{1/2} = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \left[(k+1)(x^1)^{1/2} + (n-1-k)(x^2)^{1/2}\right]^{1-2\gamma}. \tag{4.7}$$

Respectively, for an individual with type 2:

$$(x^2)^{1/2} = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} 2\left[(k(x^1)^{1/2} + (n-k)(x^2)^{1/2}\right]^{1-2\gamma}. \tag{4.8}$$

The ex-post efficient provision level only depends on the number of individuals with a certain type. For a state of the world $\theta \in \Theta$ where the number of individuals with type 1 is $0 \leq k \leq n$, the efficient provision solves

$$\max_{F \geq 0} k\frac{F^{1-\gamma}}{1-\gamma} + 2(n-k)\frac{F^{1-\gamma}}{1-\gamma} - F,$$

whose first order conditions imply that

$$F^e(\theta) = (2n-k)^{1/\gamma}. \tag{4.9}$$

On the other hand, the ex-ante optimal provision level must solve

$$\max_{F\geq 0}\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+2(n-k))\frac{F^{1-\gamma}}{1-\gamma}\right]-F,$$

and so the first order conditions yield

$$F^{EA}=\left[\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(2n-k)\right]^{1/\gamma}. \tag{4.10}$$

For a set of given parameter values, I use equations (4.7) and (4.8) to compute the individual contributions to QF, equation (4.9) to compute the ex-post optimal provision levels, and equation (4.10) to compute the ex-ante optimal provision level. In this type of game, I also divide the absolute deadweight loss to obtain a per capita measure of the inefficiency imposed by QF.

Whenever it is infeasible to calculate the values presented above directly, I employ the R package `nleqslv`, which contains algorithms to solve nonlinear equation systems using a Broyden Secant method or a full Newton method (Hasselman 2018). For each figure in the next sections, I estimated the deadweight loss at a large number of values of the parameter of interest and plotted a smoothing spline fitting these values. Also, since the relative deadweight loss measure is only properly defined when there are at least two different ex-post efficient funding levels (otherwise the denominator would be equal to zero), I omit these cases from the estimations of relative inefficiency.

The values of γ I use are those between 0.1 and 5, since, on the one hand, $\gamma = 0$ violates the hypothesis of strict concavity, and on the other hand values of γ larger than 5 are rarely used in the literature (e.g. see Huang, Milevsky, and Wang 2008; Issler and Piqueira 2000; Pasin and Vargiolu 2010). When varying the values of parameters that are not γ, I also usually present plots fixing the value of γ at 1 and 1/4. This is motivated by my analysis in section 2.3 showing that individuals behave differently under QF when $\gamma > 1/2$ or $\gamma < 1/2$.

4.2 Two Players

In this section, I use the following values for parameters as a baseline: $\alpha = 1/2$, $\beta_1 = 2$, $\beta_2 = 1$. When I am not varying one of these specific parameters, these are the values I use for them.

Note that, for this game, since we are dealing with individuals with CRRA utility functions for the public good, proposition 3.10 implies that efficiency only occurs when $\gamma = 1/2$ or when there is only a single possible equilibrium. There are three scenarios where the latter happens in my analysis, namely when $\alpha = 1$, $\alpha = 0$ or $\beta_1 = 1$. All these cases are essentially situations of complete information, as in all of them individual 1 has a certain utility function with probability one. It is also worth mentioning that, from the proof of proposition 3.10, it follows that the second player in this game can be interpreted as representing the aggregation of an arbitrary number of individuals with a single type.

I begin by varying α in figure 4.1. In both cases, the resulting inverted "U" shape has a clear intuitive explanation: the closer the game is to complete information ($\alpha = 0$ or $\alpha = 1$) the lower the deadweight loss is, and it is highest at about halfway between these values. However, the maximum of this function is not at $\alpha = 1/2$ in either case, but at $\alpha \approx 0.48$ when $\gamma = 1$ and at $\alpha \approx 0.53$ when $\gamma = 1/4$, pointing that the probability that generates the most inefficiency is related to the value of γ.

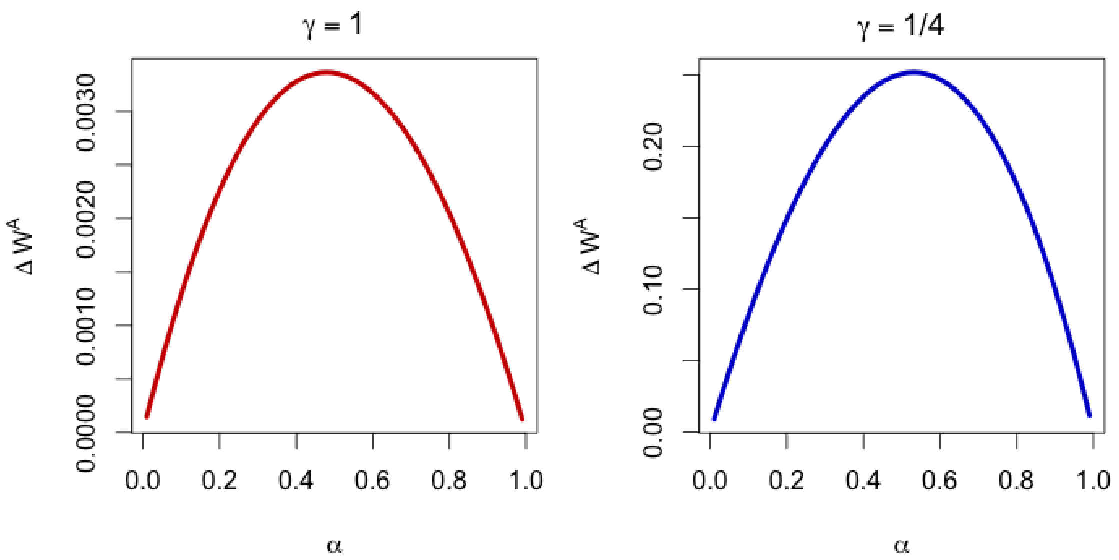

Figure 4.1: Absolute deadweight loss as a function of $\alpha \in [0, 1]$.

Now, turning to the relative deadweight loss of $\alpha \in (0, 1)$ in figure 4.2, the first noteworthy aspect is the approximately linear relationship between these variables.[1]

[1] ΔW^R is not defined for $\alpha = 0$ or $\alpha = 1$, since the ex-ante optimal provision is efficient in those

When $\gamma = 1$, the graph is downward sloping, indicating that the closer α is to 1, that is, the higher the probability that $\theta_1 = \theta_1^1$ (the type with a lower valuation of the public good), the larger the advantage of QF in comparison to the ex-ante optimal provision level. This relationship is again dependent on γ, since for $\gamma = 1/4$ the graph of this function is upward sloping, indicating that QF is relatively better for lower probabilities that $\theta_1 = \theta_1^1$. Both cases show that, for all values of $\alpha \in (0, 1)$, QF is considerably more efficient than the ex-ante optimal provision, as the relative deadweight loss ranges between 0.05 and 0.11.

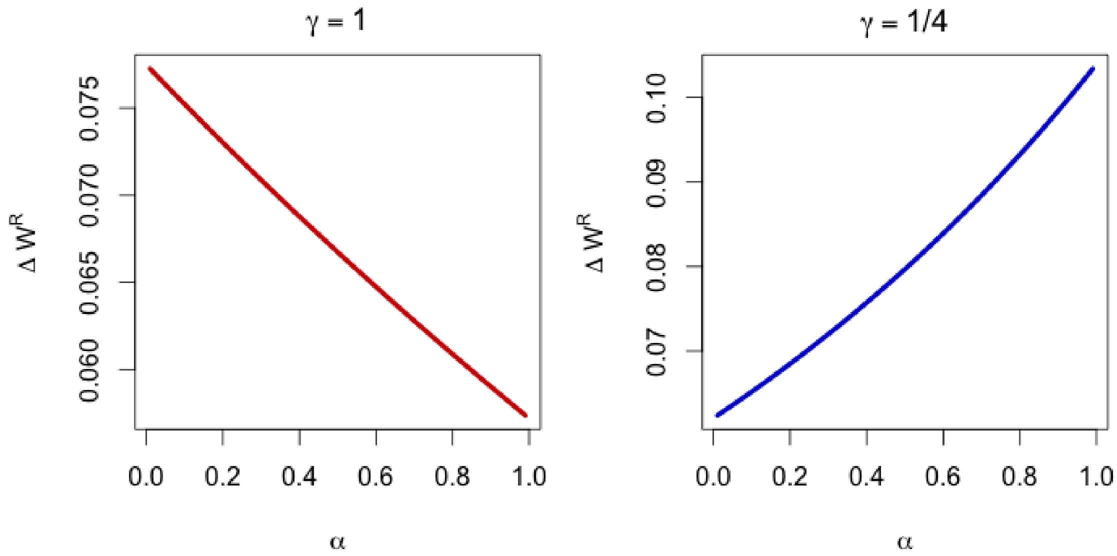

Figure 4.2: Relative deadweight loss as a function of $\alpha \in (0, 1)$.

Before I begin analyzing β_1 and β_2, I briefly explain a result that follows from my discussion in section 2.3. For an increase in β_1, the contribution of individual 1 with type θ_1^2 also increases. This is owing to the fact that a larger β_1 increases $v_1'(F; \theta_1^2)$ at every $F > 0$, so, *ceteris paribus*, only the right-hand side of equation (3.2) would increase, leading the first-order conditions to hold with strict inequality. If $\gamma > 1/2$, then by equation (4.3), individual 2's best response to this increase in expectation of individual 1's contribution is to lower his own contribution. Individual 1 would then respond to the reduction in the contribution of the other individual by *increasing* her own contribution, for both of her types. The last two steps of this process would repeat and converge to an equilibrium, resulting in individual 1 contributing more

cases.

in both states of the world and individual 2 contributing less. When $\gamma < 1/2$, an analogous reasoning applies, ultimately resulting in both individuals contributing more in both states of the world. Similarly, when the parameter that increases is β_2, this logic implies that, when $\gamma < 1/2$, all contributions are higher, and when $\gamma > 1/2$ the contributions of individual 1 in both states of the world would decrease, but that of individual 2 would increase.

I now consider changes to β_1. Figure 4.3 presents the absolute deadweight loss when varying β_1 between 0.1 and 50. As expected, the mechanism is efficient when $\beta_1 = 1$, since it is a situation of complete information. As β_1 moves away from zero, the difference between the utility functions of the types of individual 1 increases, and the fact that individual 2 faces a larger uncertainty over the contribution of individual 1 reflects in an increase of inefficiency in both directions. As I argued in the previous paragraph, when $\gamma = 1 > 1/2$ an increase in β_1 results in an equilibrium where individual 1 contributes more in both states of the world than before and individual 2 contributes less. As β_1 grows arbitrarily large, so does individual 1's contribution, and hence the second term on the right-hand side of equation (4.3) goes to zero. This results in individual 2's contribution converging to a positive value, and the decreasing rate of convergence explains the concave shape of the function to the right of $\beta_1 = 1$. To the left of $\beta_1 = 1$, as β_1 goes to 0, the same logic implies that the contribution of individual 2 grows at increasing rates, and so we observe a convex behavior. When $\gamma = 1/4 < 1/2$, an analogous reasoning shows that individual 2's contribution grows without a bound in response to increases in β_1, thus generating the convex shape observed in the figure.

Figure 4.4 presents the plots for the relative deadweight loss when varying β_1. Note that ΔW^R is not defined for $\beta_1 = 1$, as that is a situation of complete information, so the value of the graph when $\beta_1 = 1$ represents instead the limit of the function when β_1 goes to 1. In both cases, ΔW^R converges to zero as β_1 goes to infinity, which indicates that, although the deadweight loss of QF grows larger as β_1 increases, the deadweight loss of the ex-ante optimal provision grows at an even faster pace. This reflects, in part, the fact that increases in $\beta_1 > 1$ mean that information about θ_1 becomes more valuable, but cannot be obtained at all using

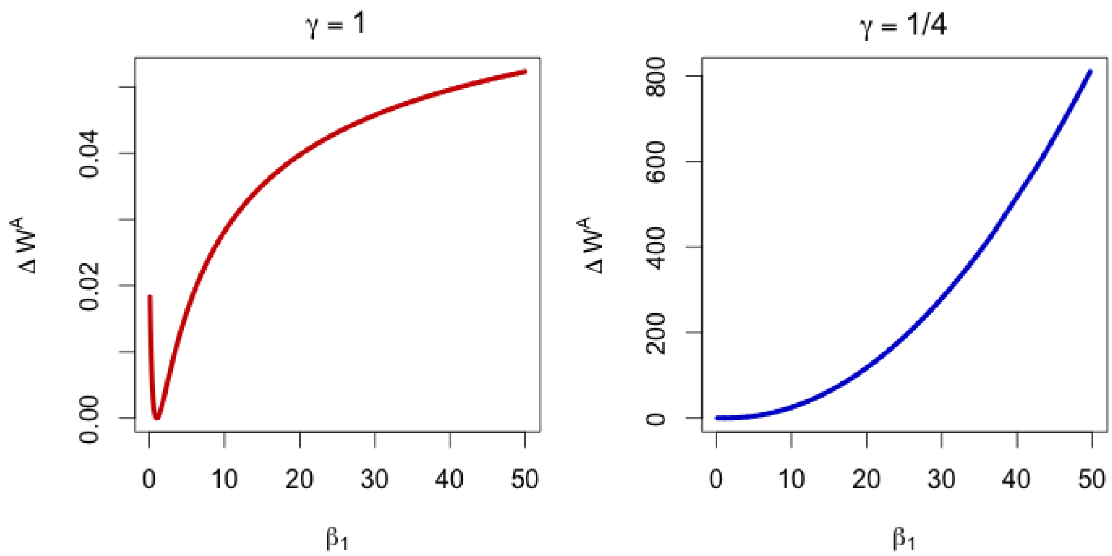

Figure 4.3: Absolute deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

the ex-ante optimal provision. However, contrarily to the expected, an increase in uncertainty when $\beta_1 < 1$ actually lowers the relative efficiency of using QF versus the ex-ante optimal provision, which is contrary to what would be expected if the aforementioned reason was the only relevant one.

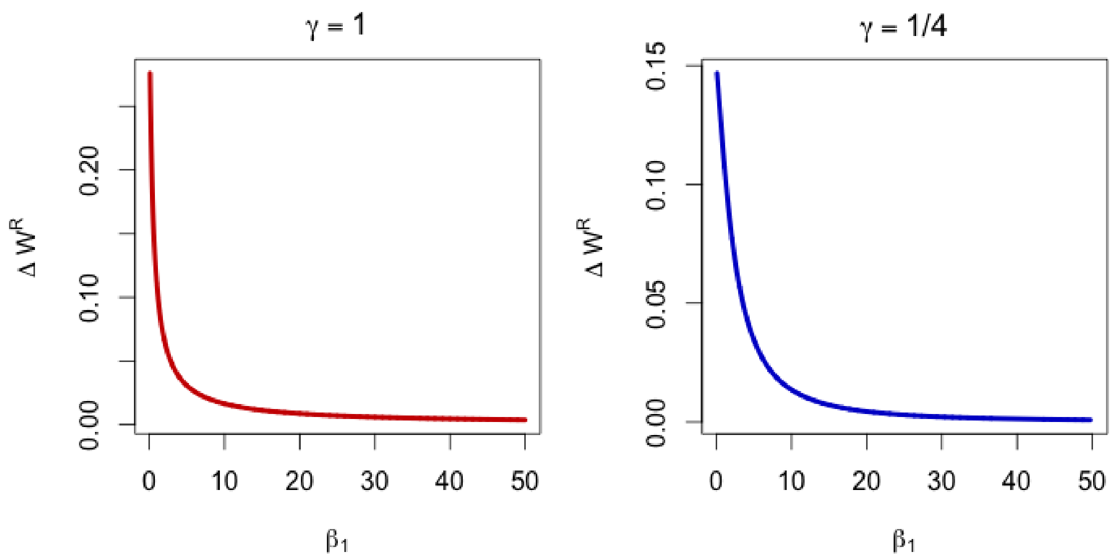

Figure 4.4: Relative deadweight loss as a function of $\beta_1 \in [0.1, 50]$.

Now turning to the analysis of changes in β_2, figure 4.5 shows the deadweight loss as a function of this parameter. We can see that, when $\gamma = 1$, inefficiency approaches zero when β_2 goes both to zero and to infinity. When β_2 approaches zero, this result follows from the fact that individual 2 values less the public good, and thus his contribution matters less to the public good. If $\beta_2 = 0$, then individual 2 does not value the public good, so he makes a contribution of zero. In response, individual 1 contributes his private valuation for the public good, and so the equilibrium of this game is efficient.

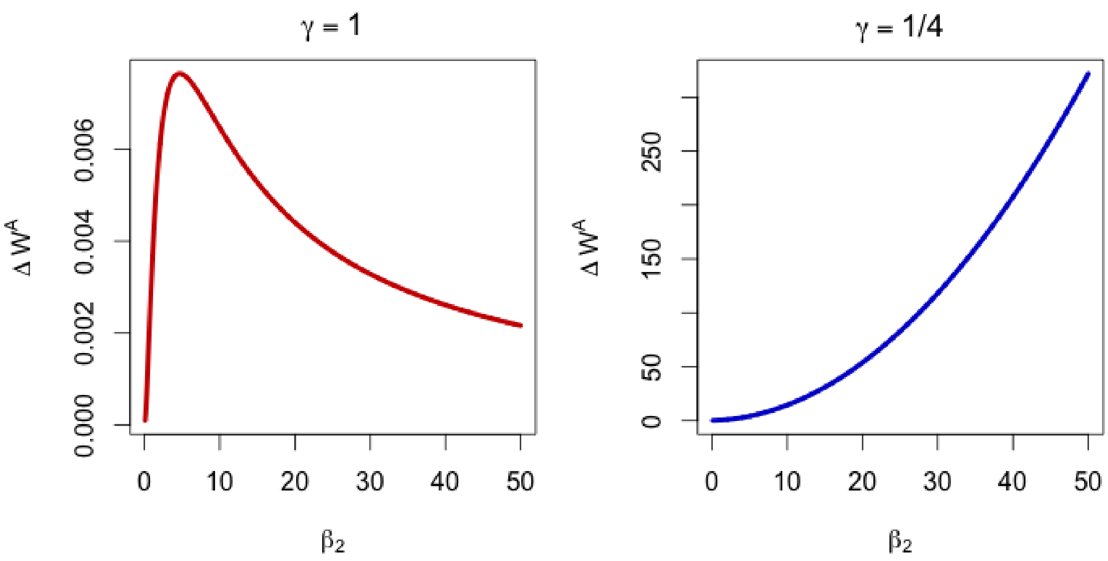

Figure 4.5: Absolute deadweight loss as a function of $\beta_2 \in [0.1, 50]$.

My previous discussion of the effect of increases in β_2 can be used to interpret the fact that the deadweight loss of QF goes to zero as β_2 goes to infinity. As mentioned earlier, an increase in β_2 when $\gamma > 1/2$ leads to an equilibrium where individual 2 contributes more and individual 1 contributes less in both states of the world. Thus, as β_2 goes to infinity, the contribution of the first individual converges to zero in both states of the world, and consequently the difference between these contributions also diminishes. This reduction in variance attenuates the problem posed by incomplete information, and hence lowers the deadweight loss of the mechanism.

When $\gamma = 1/4$, the analysis is similar. The convergence of the deadweight loss to zero when β_2 goes to zero occurs for the same reason. However, since here $\gamma > 1/2$, as the contribution of individual 2 to the public good increases, the contributions of individual 1 in both states of the world also increase. The gap between the contribution of individual 1 for her two types grows with β_2, generating a larger uncertainty for individual 2 with respect to the value of individual 1's contribution, and so the deadweight loss increases as a consequence.

The relative deadweight loss for changes in β_2 shows a similar pattern both when $\gamma = 1$ and when $\gamma = 1/4$, as seen in figure 4.6. When β_2 goes to zero, the game approaches a situation where only individual 1 contributes to the public good, and so the information contained in his contribution becomes more relevant. Under QF, this individual is able to choose his contribution, but since the ex-ante optimal provision does not use information about the state of the world, it leads to inefficiency in this scenario, and so the relative deadweight loss goes to zero. When $\beta_2 > 0$, QF is inefficient, and so ΔW^R is positive. Note that the reason presented above for the reduction in inefficiency for QF as β_2 approaches infinity in the case where $\gamma = 1$, namely that the contributions of individual 1 go to zero, is also a reason for the deadweight loss of the ex-ante optimal provision to go to zero. Hence, the convergence to zero of the absolute deadweight loss as β_2 goes to infinity for $\gamma = 1$, in figure 4.5, would not be a reason to expect the same behavior for the relative deadweight loss, which indeed does not occur, as figure 4.6 shows.

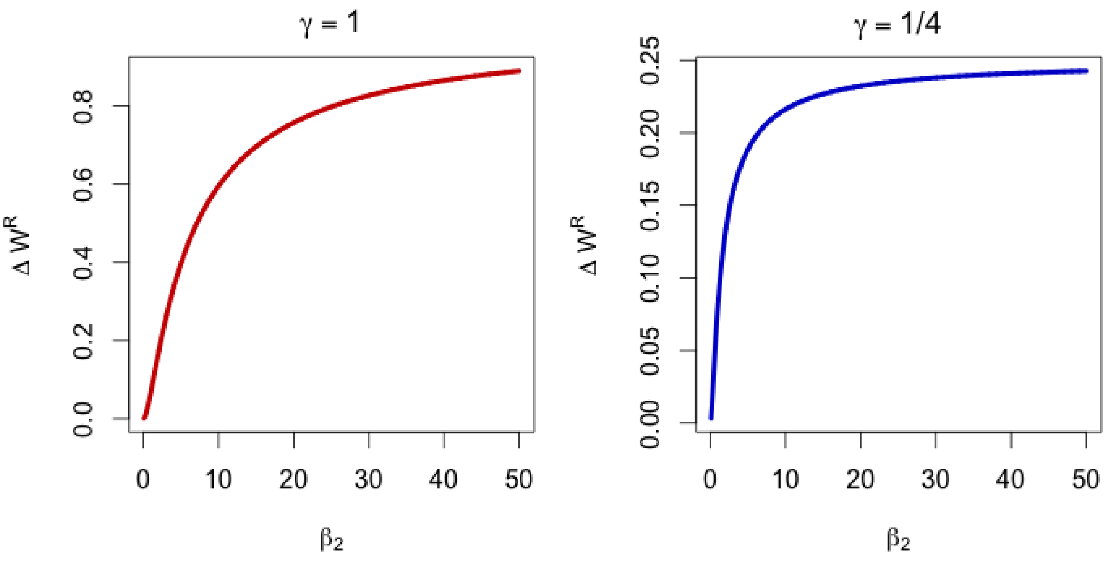

Figure 4.6: Relative deadweight loss as a function of $\beta_2 \in [0.1, 50]$.

In figure 4.6, we can also observe that the relative deadweight loss appears to grow to a value close to one when $\gamma = 1$, and to a value close to 1/4 when $\gamma = 1/4$. It is then natural to ask whether the limit of the relative deadweight loss in these plots grow for $\gamma > 1$ and, in particular, whether we could find a game for which $\Delta W^R > 1$, i.e., QF is less efficient than the ex-ante optimal provision. In figure 4.7, I present a similar plot fixing $\gamma = 1.5$, from which we can see that, indeed, there are games in which QF is less efficient than the ex-ante optimal provision. For values of β_2 close to 50, we see that the deadweight loss of QF is over 3 times larger than that of the ex-ante optimal provision, indicating that in some situations QF can be substantially worse than my benchmark provision. The value of $\gamma = 1.5$ is often used in the literature, and can even be considered a low value for this parameter (Huang, Milevsky, and Wang 2008). Additionally, as I mentioned at the beginning of this section, individual 2 here can represent an aggregation of several players, so even values of β_2 that are considerably higher than those of β_1 could represent games where there are several players with a single type and only one with multiple types. Thus, the situation presented here where QF perform worse than the ex-ante optimal provision is plausible, and should be taken into consideration when employing this mechanism.

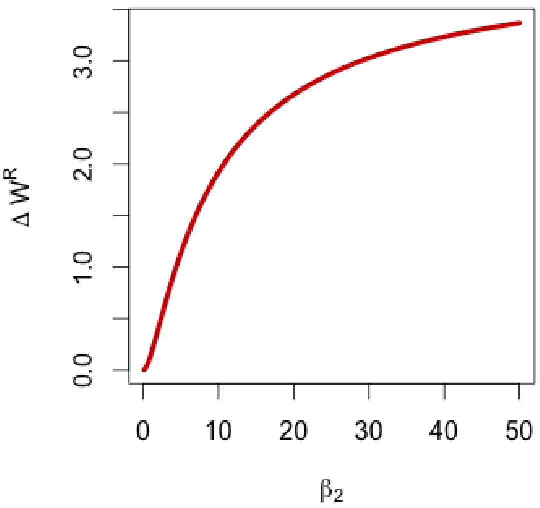

Figure 4.7: Relative deadweight loss as a function of $\beta_2 \in [0.1, 50]$, for $\gamma = 3/2$.

Lastly, I directly analyze variations of γ. Figure 4.8 presents the absolute deadweight loss as a function of γ. As we can see, inefficiency grows arbitrarily large as γ approaches zero. Equations (4.4) and (4.5) indicate that the optimal funding levels in both states of the world converge to infinity, and the same happens with difference between them, which ultimately results in the inefficiency of QF growing arbitrarily large.

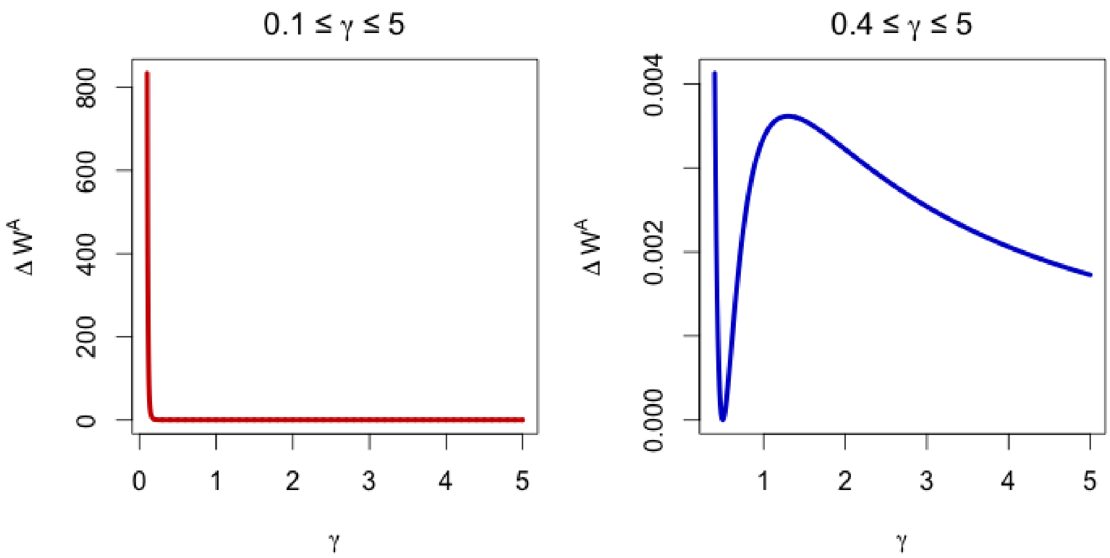

Figure 4.8: Absolute deadweight loss as a function of γ.

To analyze the behavior at larger values of γ, in the graph on the left of figure 4.8 I limit the range of values of γ to the interval $[0.4, 5]$. As proposition 3.13 proved, QF is efficient when $\gamma = 1/2$. For $\gamma > 1/2$, however, we can see that the deadweight loss of the mechanism first increases, but then starts to decrease, potentially converging to zero. One reason for the drop in inefficiency for larger values of γ is that, again by equations (4.4) and (4.5), the optimal level of provision converges to one in both states of the world as γ goes to infinity, and so the difference between the optimal provision across states of the world grows goes to zero, reducing the problem introduced by incomplete information. Figure 4.9 presents evidence favorable to this analysis, as the relative deadweight loss does not converge to zero as γ goes to infinity.[2]

In conclusion, the results of this section has several implications for the behavior of QF under incomplete information. First, changes in the value of γ significantly

[2]Although the ΔW^R could, in principle, start displaying a behavior of convergence to zero for values of γ larger than 5, that is not the case at least for values of γ as high as 100. I did not present a figure for this case here, however, since these larger values for γ are not of interest to my analysis, as justified in section 4.1.

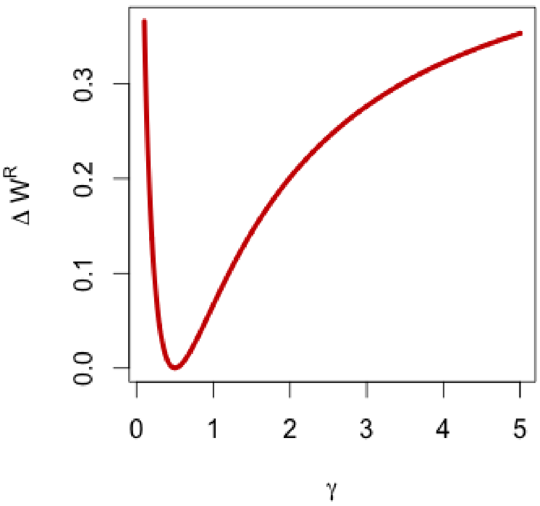

Figure 4.9: Relative deadweight loss as a function of $\gamma \in [0.1, 5]$.

alter the patterns observed in the graphs in nearly all cases. This finding indicates that an empirical understanding of how individuals respond to changes in contributions by others, be it by increasing ($\gamma < 1/2$), maintaining ($\gamma = 1/2$) or decreasing ($\gamma > 1/2$) their own contribution can generate important insights into the overall behavior of inefficiency for changes in the game. Second, we have seen that a higher variance of individual contribution increases inefficiency in absolute terms (figure 4.3), reaching particularly high levels when $\gamma < 1/2$. However, in both cases, this higher variance increases the relative efficiency of QF when compared to my benchmark model (figure 4.4), indicating that the mechanism might still be comparatively better in situations with high contribution variance. I have also shown that when an individual with a single type values more the public good, the outcome in terms of efficiency depends on the value of γ (figure 4.5), and only converges to zero when the other individual responds by diminishing his contribution, i.e., $\gamma > 1/2$. Lastly, with respect to extreme values of γ (figure 4.8), very low values cause inefficiency to increase without bound, while very high values cause inefficiency to converge to zero.

4.3 Varying Population

This section is structured in the following way. I first present and discuss the results for the total deadweight loss and for the deadweight loss per capita. My aim here is to check whether QF is able to achieve asymptotic efficiency, that is, a convergence of the absolute deadweight loss towards zero (Lalley and Weyl 2019), or checking if this convergence occurs at least at a per capita level. I then proceed to present and discuss the results for relative deadweight loss, showing in which situations QF is better than my benchmark provision for large populations.

Before presenting the results in this section, I argue that there are two underlying effects of population increases on the variance of the equilibrium level of funding. The first and most obvious effect is that a larger population increases the number of possible equilibrium values, in accordance to the first order conditions in equations (4.7) and (4.8), and so the variance of the provision level increases. This effect intensifies the problem generated by incomplete information, resulting in an increase in deadweight loss. There is, however, a second effect that takes place, which is dependent on γ. When n increases, the first order condition implies that the extra individuals make positive contributions, which would *ceteris paribus* raise the level of funding for the public good. However, in accordance to section 2.3, different values of γ result in different responses of individual contributions to changes in aggregate provision. When $\gamma > 1/2$, individuals reduce their contributions, which converge to zero as n approaches infinity. This promotes a reduction in the dispersion of contributions, and thus lowers the problem caused by incomplete information. On the other hand, when $\gamma < 1/2$, the opposite happens, and so the variance of contribution increases, contributing to an increase in inefficiency. As before, when $\gamma = 1/2$, contributions do not change in response to the extra population, so this second effect does not occur. Below, I refer to these effects as contributor quantity effect and contribution dispersion effect, respectively.

In an attempt to isolate the contributor quantity effect I present the deadweight loss for the case where $\gamma = 0.499$. The reason for doing so is that it should be close enough to $\gamma = 1/2$ for the contribution dispersion effect to be comparatively small, while also being different from $\gamma = 1/2$ and thus having a non-zero deadweight loss for $n \geq 2$. The resulting plots are very similar to those using $\gamma = 0.501$, which indicate that this degree of approximation is sufficient for the contributor quantity effect to dominate. As we can see in figure 4.10, the deadweight loss in this case is approximately linear.

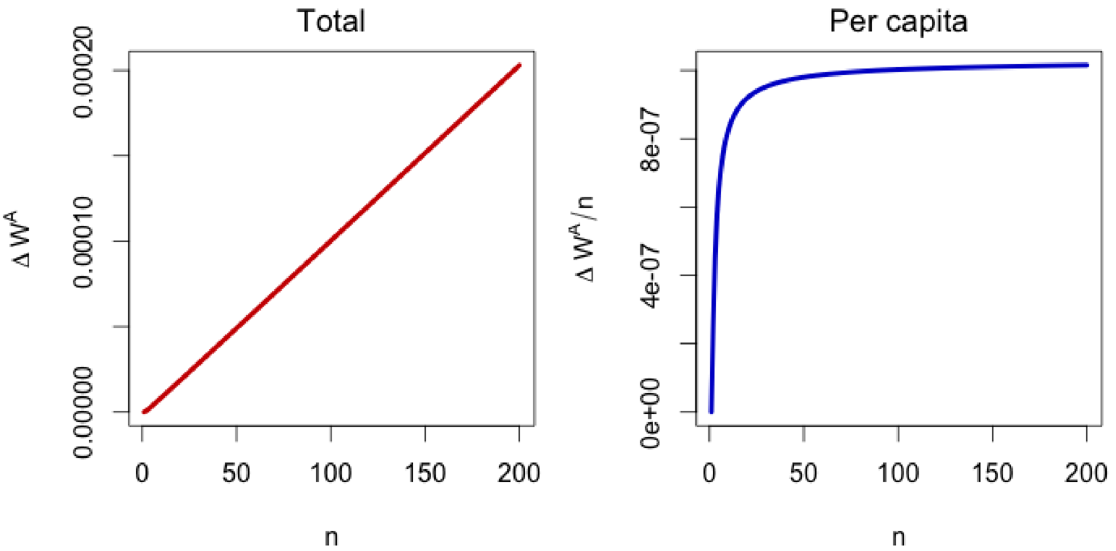

Figure 4.10: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 0.499$.

Thus, for the cases I present below where γ is more distant from $1/2$, based on my earlier argument we should expect the contribution dispersion effect to reduce the rate of growth promoted by the contributor quantity effect, and the contrary would happen when $\gamma < 1/2$. In figure 4.10 we can also see that the deadweight loss per capita appears to converge to a positive number, and so the negative sign of the dispersion effect when $\gamma > 1/2$ should make it decrease and converge to zero, while a positive sign of this effect when $\gamma < 1/2$ should make deadweight loss per capita increase at increasing rates with population.

Figure 4.11 then shows the absolute deadweight loss for $\gamma = 1$. As we can see, here the contribution dispersion effect does lower the rate of growth of inefficiency. The magnitude of the effect appears to be large enough to make the deadweight loss converge, however it is not large enough to make it converge to zero. Hence, in this case, we can conclude that QF is not asymptotically efficient. When we divide this measure by population size, however, the convergence of the deadweight loss to a strictly positive number causes the per capita measure to converge to zero. Interestingly, the peak in this figure occurs at $n = 4$, showing that the contributor quantity effect is strong enough for $2 \leq n \leq 4$ to make the deadweight loss per capita increase with population.

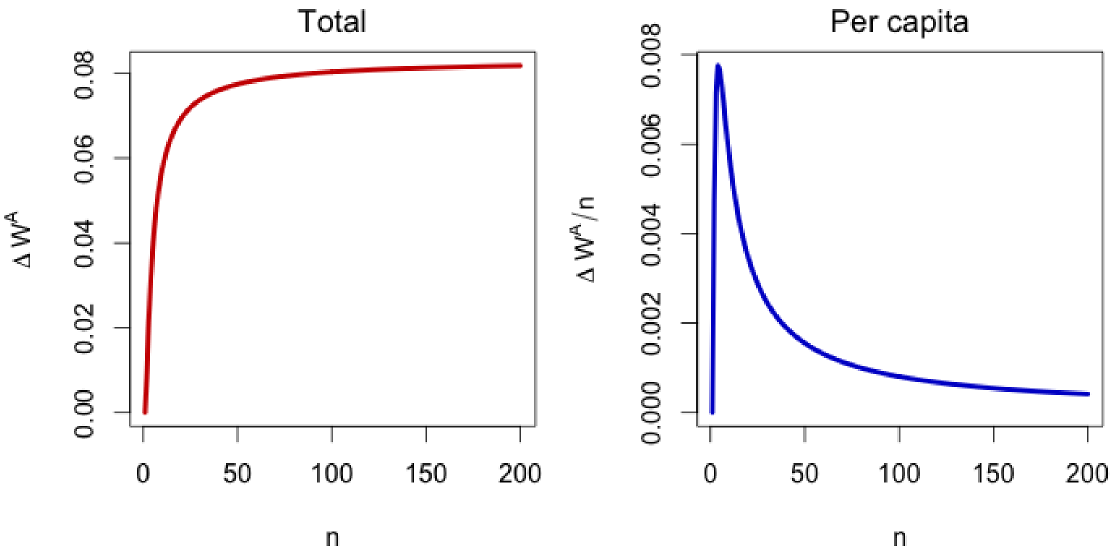

Figure 4.11: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 1$.

Now, in figure 4.12 I set $\gamma = 1/4$. The combination of a positive contributor quantity effect and a positive contribution dispersion effect makes the total absolute deadweight loss grow at increasing rates, reaching the largest order of magnitude of all figures presented in this work. In this case, the contribution dispersion effect is strong enough that even the absolute deadweight loss per capita loss is a convex function of n.

Figure 4.13 shows the results for $\gamma = 3/2$. In comparison with the case where $\gamma = 1$, the contribution dispersion effect is stronger here, to the extent that the absolute deadweight loss starts to decrease after $n = 16$, slowly converging to zero and resulting in asymptotic efficiency. Naturally, the per capita effect also converges to zero, doing so at an even faster rate.

I now evaluate of the relative deadweight loss in figure 4.14. The first noteworthy aspect of these graphs is that they all have the same shape, growing at decreasing rates and then apparently converging to a positive value. This pattern highlights the fact that the advantage of QF lies in letting individuals choose their contri-

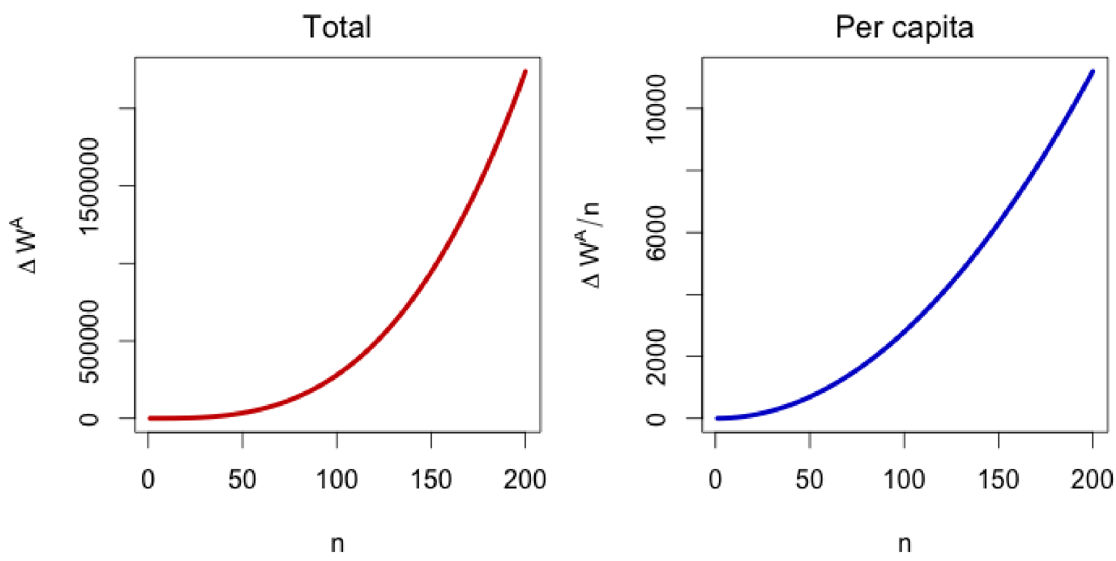

Figure 4.12: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 1/4$.

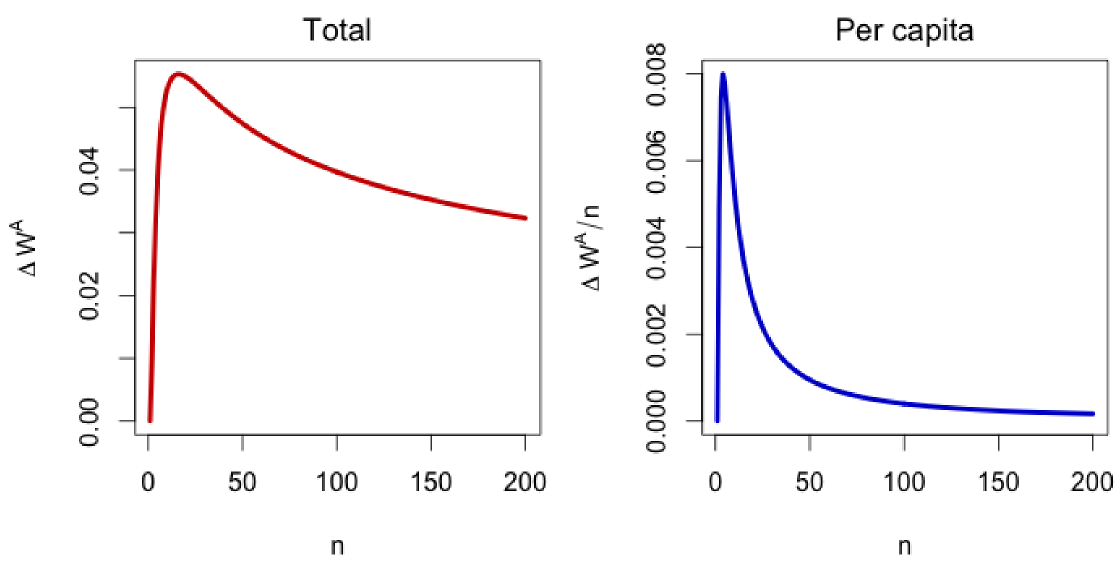

Figure 4.13: Absolute deadweight loss as a function of $1 \leq n \leq 200$, for $\gamma = 3/2$.

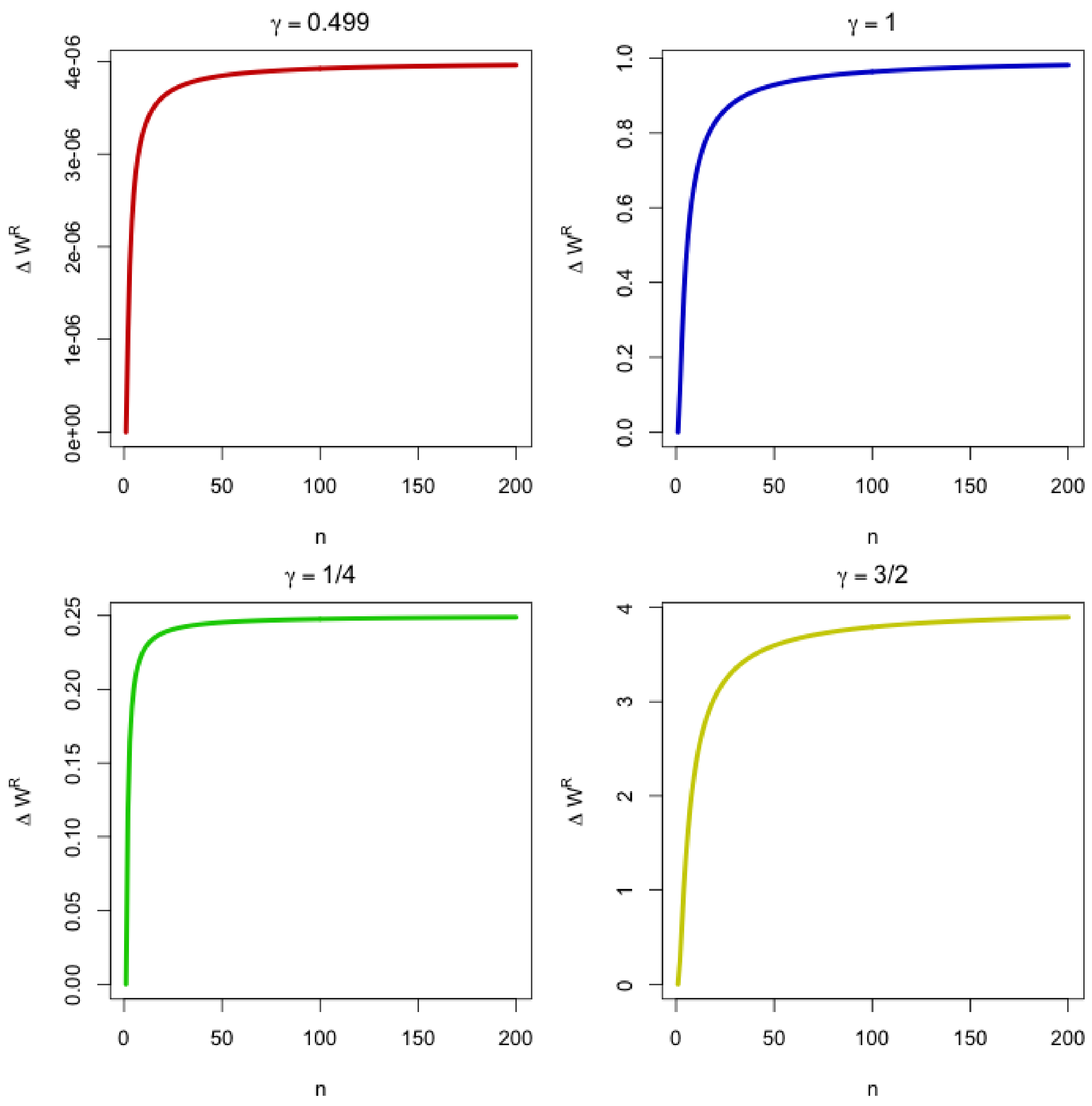

Figure 4.14: Relative deadweight loss as a function of $1 \leq n \leq 200$.

butions using their private information , but a larger population diminishes the relative importance of the information possessed by each individual, thus lowering the comparative efficiency of this mechanism.

With regards to the level of relative inefficiency, for $\gamma = 1$ this measure appears to converge to 1 when n goes to infinity, indicating that QF is asymptotically equivalent to the ex-ante provision in terms of efficiency. When $\gamma = 1/4$, although the absolute deadweight loss of QF can get considerably high for large populations, the ex-ante optimal provision has an even higher expected inefficiency, to the extent that QF is considerably more efficient even for larger populations. Lastly, if $\gamma = 3/2$, we can see that QF becomes more inefficient than using the ex-ante optimal provision. This last result is similar to that presented for increases in β_2 in the previous section, and reinforces the idea that QF can be relatively less useful for cases where $\gamma > 1$.

The results presented in this section suggest that larger population sizes can pose a significant challenge to QF. I have shown that asymptotic efficiency does not occur when $\gamma = 1/4$, $\gamma = 0.499$ or $\gamma = 1$, and only in the last of these does per capita inefficiency go to zero. When $\gamma = 3/2$, the mechanism does appear to be asymptotically efficient, but then it also becomes worse than providing the ex-ante optimal level for larger populations sizes. Being less efficient than the ex-ante optimal provision is particularly problematic, given that providing this benchmark level of funding is much simpler to implement than adequately running the QF mechanism. Since values of $\gamma > 1$ are frequently used in the literature (Huang, Milevsky, and Wang 2008; Issler and Piqueira 2000; Pasin and Vargiolu 2010), the class of situations where QF might be most useful are those where the value of this parameter is relatively low. Also, due to the problems that occur even when $\gamma < 1$, the best usage cases for QF might be situations where the population size is small and $\gamma < 1$, or those where γ is close to $1/2$ for any population size.

5. Conclusion

Quadratic funding is a public good provision mechanism with several interesting properties, both from a theoretical and from an empirical perspective. In the present work, I have advanced the theoretical understanding of this mechanism, especially in, but not limited to, a setting of incomplete information.

First, in chapter 2, I started by showing that, under a reasonably weak set of hypotheses, there exists a unique efficient provision level for the public good in a game of complete information. I then proceeded to show the incompleteness of the efficiency result presented by Buterin, Hitzig, and Weyl (2019), and subsequently extended it to show that the mechanism is also efficient in cases where no individual would privately fund the mechanism. I also showed that the mechanism has at most two equilibria, and that when two equilibria exist, one of them is not optimal. Lastly, I showed that when the individual utilities for the public good take the form of a CRRA utility function, the value of the relative risk aversion coefficient indicates how the individual responds to changes in the contributions of other individuals.

In chapter 3, I adapted the previous framework to a setting of incomplete information, and proved several relevant theoretical results for this case. After showing existence results for both the efficient provision levels for the public good and for equilibria of the quadratic funding mechanism, I have proved that the efficiency of the mechanism under complete information fails to generalize here in several different situations. If the optimal provision is zero with positive probability, then the mechanism is only efficient when this probability is equal to one, indicating that the mechanism cannot provide the public good only when it is efficient to do so. Even when the public good should be provided with certainty, I show that efficiency follows almost exclusively on situations where individuals would like to make the same contribution in all possible states of the world conditional on their types, and this condition fails to hold in most cases. I exemplify this lack of generality by showing that, if the individual utility functions for the public good are CRRA functions with the same coefficient of risk aversion, then efficiency only follows when this coefficient is equal to a half.

In chapter 4, I used the theory previously developed as a basis for doing quantitative estimates of quadratic funding's inefficiency. The class of utility functions for the public good chosen for this analysis was that of CRRA functions, due to the previously mentioned importance of the risk aversion coefficient. I developed two measures of inefficiency, one that assesses the expected monetary loss of the mechanism, and another that takes the ratio of expected inefficiency of quadratic funding with a benchmark provided by the expected inefficiency of the ex-ante efficient level of the public good. Using two types of games, I analyzed how changes in the values of parameters influence inefficiency, leading to several useful results. With regards to the number of players, larger population sizes lead to explosive growth of inefficiency when the risk aversion coefficient is lower than a half, and leads the mechanism to perform comparatively worse than my benchmark measure when it is higher than the unit. For small populations, analogous results apply if a player that has incomplete information values the public good highly. An increased expected variance of individual contribution has also been shown to increase expected inefficiency, although the rate at which inefficiency grows depends on the coefficient of risk aversion. Another noteworthy result is that inefficiency seems to be a continuous function of the parameters I analyzed, so situations with a relative risk aversion coefficient very close to a half have low inefficiency.

These results bring important implications both in theoretical and in practical terms. On the theoretical side, using my interpretation of public good provision mechanisms as production functions, my findings demonstrate that no element of the class of CES functions can provide a socially optimal level for public goods in the general case. I have shown that a sufficient condition for the optimality of quadratic funding under incomplete information is that, conditional on her type, an individual would like to make the same contribution in every possible state of the world. This result indicates that trying to find mechanisms for which this condition is more broadly true, or perhaps universally true, is a promising path for finding efficient mechanisms for public good provision. The measure of relative inefficiency I have proposed is a good proxy for how well a mechanism is using information that is not common knowledge in the game, and the idea behind it can be useful for evaluating incomplete information games. Lastly, the results presented here can serve as a basis for the development of variations of quadratic funding that attempt to circumvent the informational problems it faces.

On the practical side, the theoretical and quantitative results presented here provide a point of reference for what to expect from quadratic funding in terms of efficiency. In particular, situations where individuals do little in response to changes in the contributions of others might indicate that a CRRA utility with relative risk aversion coefficient of a half is a good representation of individual utilities, and so the efficiency loss imposed by quadratic funding is low. Games where the number of players is particularly large, or where the range of possible valuations of the public good is broad, are particularly problematic. Finally, and more generally, my results show that adapting the context where quadratic funding is used so as to make it more similar to a complete information scenario can generate large efficiency gains.

A. Proofs

Proof of Proposition 2.8. Let the function $v : \mathbb{R}_+ \to \mathbb{R}$ be defined as $v(F) = \sum_{i=1}^{n} v_i(F)$. With the assumptions given, we have that v is C^1, strictly concave, and $\lim_{F\to\infty} v'(F) = 0$. We then have two cases: $v'(0) \leq 1$ or $v'(0) > 1$. In the first case, it follows from definition 2.4 that $F = 0$ is optimal. Furthermore, since v is strictly concave, we have that $v'(F) < 1$ for all $F > 0$ and thus, again from definition 2.4, there can be no socially optimal provision $F > 0$. Thus, $F^e = 0$ is the unique efficient provision.

Now suppose that $v'(0) > 1$. It follows by definition that $F = 0$ is not optimal. As $\lim_{F\to\infty} v'(F) = 0$, we have that there exists $A \in \mathbb{R}$ such that $v'(A) < 1$. Thus, since v' is continuous, $v'(0) > 1$ and $v'(A) < 1$, it follows from the intermediate value theorem that there exists $0 < B < A$ such that $v'(B) = 1$. Additionally, since v' is strictly decreasing (thus injective), we have that $v'(F) \neq 1$ for all $F \neq B$. Therefore, $F^e = B$ is the unique efficient provision.

We then have that in both cases there is a unique efficient provision $F^e \geq 0$, as desired. □

Proof of Proposition 2.10. By proposition 2.8, we know that the hypotheses adopted here guarantee that there is a unique socially optimal provision $F^e \geq 0$. There are two cases: $F^e = 0$ or $F^e > 0$. First, suppose we have $F^e = 0$. We want to show that $\mathbf{0}$ is an equilibrium for Φ^{QF}. Consider the problem faced by some individual $i \in \mathcal{N}$

when all other individuals contribute zero to the mechanism:

$$\max_{c_i \geq 0} v_i \left(\left[c_i^{1/2} + 0 \right]^2 \right) - c_i.$$

Which can be written as

$$\max_{c_i \geq 0} v_i \left(c_i \right) - c_i.$$

Thus, the first order condition for the individual i is that $v_i'(c_i) \leq 1$, with equality holding when $c_i > 0$. But note that, since $F^e = 0$, we have from definition 2.4 that $\sum_{j=1}^n v_j'(0) \leq 1$. In particular, since v_j is increasing for all $j \in \mathcal{N}$, this implies that $v_i'(0) \leq 1$. Thus, $c_i = 0$ satisfies the first order condition for i. But since the choice of i was arbitrary, we have that $\mathbf{0}$ is an equilibrium for Φ^{QF}.

Now, suppose $F^e > 0$. For all $i \in \mathcal{N}$, let

$$c_i = \left(v_i'(F^e) \cdot (F^e)^{1/2} \right)^2. \tag{A.1}$$

Rearranging, we obtain

$$v_i'(F^e) = \frac{(c_i)^{1/2}}{(F^e)^{1/2}}, \tag{A.2}$$

which is precisely the first order condition found in equation (2.1) for the individual optimization problem. We can thus conclude that the vector $\mathbf{c} = (c_1, \ldots, c_n)$ as defined by equation (A.1) is an equilibrium of QF and, by proposition 2.9, that its provision is optimal.

Thus, in all cases, there exists an equilibrium allocation $\mathbf{c}^*$ such that $\Phi^{QF}(\mathbf{c}^*) = F^e$, as we wanted to show.

□

Proof of Lemma 2.11. Suppose by contradiction that, for all $i \in \mathcal{N}$, we have that v_i is C^1 and strictly increasing, and that there exists an equilibrium strategy profile $\mathbf{c} \neq \mathbf{0}$ such that $c_j = 0$ for some $j \in \mathcal{N}$. The first order condition for j is given by

$$\frac{v_j'(\Phi^{QF}(\mathbf{c})) \cdot \left[\sum_{i=1}^n c_i^{1/2} \right]}{c_j^{1/2}} \leq 1. \tag{A.3}$$

But note that, since the aggregate contribution is strictly positive, and since v_j is strictly increasing, $c_j = 0$ is dividing a strictly positive number. Hence, the inequality in (A.3) cannot hold, implying that $c_j = 0$ does not satisfy the first order condition for j. This contradiction completes the proof. □

Proof of Proposition 2.12. First, suppose that zero is not the efficient provision, i.e., $\sum_{i=1}^{n} v_i'(0) > 1$, and also that $v_i'(0) \leq 1$ for all $i \in \mathcal{N}$. Now, consider the problem that some individual $i \in \mathcal{N}$ faces when every other individual contributes zero to the public good, that is, $c_j = 0$ for all $j \neq i$. As we have seen in the proof of proposition 2.10, this problem can be written as

$$\max_{c_i \geq 0} v_i\left(c_i\right) - c_i. \tag{A.4}$$

The first order condition is thus that $v_i'(c_i) \leq 1$, with equality holding when $c_i > 0$. But since $v_i'(0) \leq 1$, we have that $c_i^* = 0$ is a solution to this problem. Since the choice of i was arbitrary, we have that $\mathbf{0}$ is a Nash equilibrium of this game, which is not optimal by hypothesis.

Now, suppose by contradiction that there exists an inefficient equilibrium, and that it is not true that zero is not an efficient provision and $v_i'(0) \leq 1$ for all $i \in \mathcal{N}$. There are two cases: zero is an efficient provision, or there exists $i \in \mathcal{N}$ such that $v_i'(0) > 1$. If zero is an efficient provision, then it follows from lemma 2.11 that the inefficient equilibrium is interior. On the other hand, suppose that there exists $i \in \mathcal{N}$ such that $v_i'(0) > 1$. Then, if all $j \neq i$ contribute zero to the public good, the problem for i is, as in (A.4),

$$\max_{c_i \geq 0} v_i\left(c_i\right) - c_i.$$

But note that, since $v_i'(0) > 1$, $c_i = 0$ does not satisfy the first order condition $v_i'(c_i) \leq 1$. Hence, $\mathbf{0}$ is not an equilibrium. But then, again by lemma 2.11, the inefficient equilibrium must be interior. Therefore, in all cases, the inefficient equilibria must be interior. But this contradicts proposition 2.9. This contradiction completes the proof. □

Proof of Proposition 2.14. Without loss of generality, we index the individual with a CRRA utility function for the public good by the number 1. Let $v_1(F) = \beta F^{1-\gamma}/1-\gamma$, with $\beta > 0$, be this individual's utility function for public good, and let $BR_1(\overline{\mathbf{c}}_{-1}) = \overline{c}_1$ and $BR_1(\mathbf{c}_{-1}) = c_1$ be the individual's best responses to two different profiles of contributions for the other individuals in such a way that

$$\sum_{j=2}^{n} \overline{c}_j^{1/2} < \sum_{j=2}^{n} c_j^{1/2}. \tag{A.5}$$

We want to show that $c_1 \leq \overline{c}_1$ if, and only if, $\gamma \geq 1/2$.

The first order condition for the individual's optimization problem in both situation yields

$$\overline{c}_1^{1/2} = \beta \left(\sum_{j=1}^{n} \overline{c}_j^{1/2} \right)^{1-2\gamma}, \tag{A.6}$$

$$c_1^{1/2} = \beta \left(\sum_{j=1}^{n} c_j^{1/2} \right)^{1-2\gamma}. \tag{A.7}$$

Thus, the individual contributions to the public good in the scenarios above are, respectively, the zeros of the functions $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ defined as

$$f(x) = \beta \left(x^{1/2} + \sum_{j=2}^{n} \overline{c}_j^{1/2} \right)^{1-2\gamma} - x^{1/2},$$

$$g(x) = \beta \left(x^{1/2} + \sum_{j=2}^{n} c_j^{1/2} \right)^{1-2\gamma} - x^{1/2}.$$

In the case where $\gamma = 1/2$, the claim follows immediately from the first order conditions. Thus, there are two cases left to be demonstrated, when $\gamma > 1/2$ and when $\gamma < 1/2$.

First, suppose that $\gamma > 1/2$, i.e., $1 - 2\gamma < 0$. Equations (A.5) and (A.6) imply that $g(\overline{c}_1) < f(\overline{c}_1) = 0$. Since $g(0) > 0$ and g is continuous, the intermediate value theorem guarantees there exists $x^* \in (0, \overline{c}_1)$ such that $g(x^*) = 0$. Thus, $c_i = x^* < \overline{c}_1$.

Now, suppose that $\gamma < 1/2$, that is, $1 - 2\gamma > 0$. From equations (A.5) and (A.7) we have that $f(c_1) < g(c_1) = 0$. If $\sum_{j=2}^{m} \overline{c}_j^{1/2} > 0$ it then follows that $f(0) > 0$, and since f is continuous, then the intermediate value theorem guarantees the existence of $x^* \in (0, c_1)$ such that $f(x^*) = 0$. On the other hand, if $\sum_{j=2}^{n} \overline{c}_j^{1/2} = 0$, we have that $f'(0) = \infty$, and thus there exists a sufficiently small $\varepsilon > 0$ (we can assume that $\varepsilon < c_1$) such that $f(\varepsilon) > 0$. Thus, again by the intermediate value theorem there exists $x^* \in (\varepsilon, c_1)$ such that $f(x^*) = 0$. Therefore, in both cases we have $\overline{c}_1 = x^* < c_1$. This completes the proof. □

Proof of Proposition 3.5. For some $\theta \in \Theta$, let $v : \mathbb{R}_+ \times \Theta \to \mathbb{R}$ be defined by $v(F; \theta) = \sum_{i=1}^{n} v_i(F; \theta_i)$. We can show, by an analogous argument to that presented in the proof of proposition 2.8, that there is a unique efficient funding $F \geq 0$ for this state of the world $\theta \in \Theta$. Consequently, we can construct a unique function $F^e : \Theta \to \mathbb{R}_+$ that maps each state of the world to its efficient funding. $\square$

Proof of Lemma 3.6. Without loss of generality, I show that the claim holds for the best response of the individual indexed by 1. Let some $\theta_1 \in \Theta_1$ be the type of this individual, and suppose by contradiction that there exist $a, b \in \mathbb{R}_+$, $a \neq b$, which are both best responses to some $\mathbf{c}_{-i}$. Again without loss of generality, suppose $a < b$, and let $\varepsilon = b^{1/2} - a^{1/2} > 0$. From the first order conditions, we have

$$\begin{aligned}
a^{1/2} &= E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right] \\
&= a^{1/2} E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \\
&\quad + E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right].
\end{aligned}$$

Note that, since the term in the third line of the above expression is nonnegative, for the equality to hold we must have that $E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \leq 1$. Thus, since $b > a$ and v_1' is strictly decreasing, we have that

$$\begin{aligned}
&E\left[v_1' \left(\left[b^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right] \\
&\quad < E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \cdot \left[\sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right] \middle| \theta_i \right],
\end{aligned}$$

and

$$\begin{aligned}
&b^{1/2} E\left[v_1' \left(\left[b^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] \\
&\quad < a^{1/2} E\left[v_1' \left(\left[a^{1/2} + \sum_{i=2}^{n} (c_i(\theta_i))^{1/2} \right]^2 ; \theta_i \right) \middle| \theta_i \right] + \varepsilon.
\end{aligned}$$

Thus, adding the inequalities and using the fact that a and b satisfy the first order conditions, it follows that $a^{1/2} + \varepsilon > b^{1/2}$. But this contradicts the definition of ε. This contradiction completes the proof. $\square$

Proof of Lemma 3.7. Suppose that, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$ we have that $v_i(\cdot;\theta_i)$ is strictly concave and $\lim_{F\to\infty} v_i'(F;\theta_i) = 0$. We can then define, for each $i \in \mathcal{N}$, a function $f_i : \Theta_i \to \mathbb{R}_+$ that maps each θ_i to some $f(\theta_i) > 0$ such that $v_i'(f(\theta_i);\theta_i) < 1/n$. Now, letting $A := \max\{f_i(\theta_i); i \in \mathcal{N}, \theta_i \in \Theta_i\}$, by the hypothesis of strict concavity it follows that $v_i'(A;\theta_i) < 1/n$, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$. Now, suppose by contradiction that, for some $j \in \mathcal{N}$, we have that $c_i(\theta_i) \in [0, A]$ for all $i \neq j$ and all $\theta_i \in \Theta_i$, but that there exists some $k > A$ which is a best response for j with some type $\theta_j \in \Theta_j$. Since $k > 0$, the first order conditions for j (see equation 3.2), imply that

$$\begin{aligned}
k^{1/2} &= E\left[v_j'\left(\left[k^{1/2} + \sum_{i\neq j}(c_i(\theta_i))^{1/2}\right]^2;\theta_j\right)\cdot\left[k^{1/2} + \sum_{i\neq j}(c_i(\theta_i))^{1/2}\right]\,\middle|\,\theta_j\right]\\
&< \frac{1}{n}E\left[k^{1/2} + \sum_{i\neq j}(c_i(\theta_i))^{1/2}\,\middle|\,\theta_j\right]\\
&< \frac{1}{n}n\cdot k^{1/2},
\end{aligned}$$

where the first inequality follows from $v_j'(A;\theta_j) < 1/n$ and $v_j(\cdot;\theta_j)$ being strictly concave, and the second inequality follows from $c_i(\theta_i) \in [0, A]$ for all $i \neq j$. We then have that $k < k$, a contradiction. This contradiction completes the proof. □

Proof of Proposition 3.8. For each individual $i \in \mathcal{N}$, let $|\Theta_i| = L_i$. By restricting the set of possible contribution choices for each individual to the interval $[0, A]$, using the $A > 0$ specified in lemma 3.7, we can write the problem for some individual i with type θ_i as

$$\max_{c_i\in[0,A]} E\left[v_i\left(\Phi^{QF}(c_i, \mathbf{c}_{-i}(\theta));\theta_i\right)\,\middle|\,\theta_i\right] - c_i.$$

Note that, since v_i and Φ^{QF} are continuous on their arguments, the objective function is continuous on all contributions. The restriction correspondence is constant and equal to the interval $[0, A]$, and thus satisfies the properties of upper and lower hemicontinuity, and we also have that $[0, A]$ is compact and non-empty. Thus, by the maximum theorem (Ok 2007), we have that the best response correspondence for i is upper hemicontinuous and it is always at least single-valued. Additionally, by lemma 3.6, we have that this best response correspondence is at most single-valued, which implies that this correspondence is, in particular, a continuous function. Hence, since every individual has a continuous best response function, the aggregate best response function $BR : [0, A]^{\sum_{i=1}^n L_i} \to [0, A]^{\sum_{i=1}^n L_i}$ is also continuous. Finally, since $[0, A]^{\sum_{i=1}^n L_i}$ is a compact and non-empty subset of $\mathbb{R}^{\sum_{i=1}^n L_i}$, the Brouwer fixed point theorem (Ok 2007) allows us to conclude that BR has a fixed point. Thus, we can conclude that there exists an equilibrium for QF. □

Proof of Proposition 3.9. First, suppose that $F^e(\theta) = 0$ for all $\theta \in \Theta$. We want to show that the function $\mathbf{c} : \Theta \to \mathbb{R}^n_+$ defined by $c(\theta) = 0$ for all θ is an equilibrium for QF. Suppose that, for some $i \in \mathcal{N}$ we have that all other individuals play the strategy profile $\mathbf{c}_{-i} : \Theta_{-i} \to \mathbb{R}^n_+$ defined by $\mathbf{c}_{-i}(\theta_{-i}) = 0$. We need to show that $c_i : \Theta_i \to \mathbb{R}_+$ defined by $c_i(\theta_i) = 0$ for all $\theta_i \in \Theta_i$ is a best response. The problem faced by individual i with some type $\theta_i \in \Theta_i$ is given by

$$\max_{c_i(\theta_i)\geq 0} E\left[v_i\left(\Phi^{QF}(c_i(\theta_i), \mathbf{c}_{-i}(\theta)); \theta_i\right) \,\middle|\, \theta_i\right] - c_i(\theta_i).$$

But note that, since $\mathbf{c}_{-i}(\theta_{-i}) = 0$ for all $\theta_{-i} \in \Theta_{-i}$, using the definition of QF we can rewrite the problem above as

$$\max_{c_i(\theta_i)\geq 0} v_i\left(c_i(\theta_i); \theta_i\right) - c_i(\theta_i),$$

whose first order conditions are

$$v_i'(c_i(\theta_i); \theta_i) \leq 1,$$

with equality holding when $c_i(\theta_i) > 0$. But note that, since $F^e(\theta) = 0$ for all $\theta \in \Theta$, the definition implies that $\sum_{j=1}^n v_j'(0; \theta_j) \leq 1$ and so, since $v_i(\cdot; \theta_i)$ is increasing, it follows that $v_i'(0; \theta_i) \leq 1$. Thus, $c_i(\theta_i) = 0$ satisfies the first order conditions for i, as we wanted to show.

Now, suppose by contradiction that there exist $\theta' \in \Theta$ for which $F^e(\theta') = 0$, and $\theta'' \in \Theta$ for which $F^e(\theta'') > 0$, and that QF is efficient. In the state of the world θ'', note that the first order conditions for some individual i,

$$E\left[v_i'(F^*(\theta)); \theta_i'') \cdot \left(\frac{F^*(\theta)}{c_i^*(\theta_i'')}\right)^{1/2} \,\middle|\, \theta_i''\right] \leq 1, \tag{A.8}$$

cannot hold for $c_i^*(\theta_i'') = 0$, since $\Pr(\theta'') > 0$ by the hypothesis of efficiency we have that $F^*(\theta'') = F^e(\theta'') > 0$, and $v_i'(F^*(\theta''); \theta_i'') > 0$, so $c_i^*(\theta_i'')$ is the denominator of a strictly positive number. Hence, $\mathbf{c}^*(\theta'') \gg \mathbf{0}$. On the other hand, in the state of the world θ, we have that the first order conditions for some individual i are given by

$$E\left[v_i'(F^*(\theta)); \theta_i') \cdot \left(\frac{F^*(\theta)}{c_i^*(\theta_i')} \right)^{1/2} \middle| \theta_i' \right] \leq 1, \tag{A.9}$$

with equality holding when $c_i^*(\theta_i') > 0$. But since $\Pr(\theta_i', \theta_{-i}'') > 0$, and $\mathbf{c}_{-i}^*(\theta_{-i}'') \gg \mathbf{0}$, we have that $c_i^*(\theta_i') = 0$ cannot satisfy the first order conditions, as it appears on the denominator of an expression with a strictly positive numerator. Hence, $c_i^*(\theta_i') > 0$. But then $\Phi^{QF}(\mathbf{c}(\theta')) > 0 = F^e(\theta')$, a contradiction to the efficiency of QF. This contradiction completes the proof. □

Proof of Proposition 3.10. Without loss of generality, let the agent with multiple types be indexed by 1. First, suppose that QF is efficient, that is, there exists an equilibrium strategy profile $\mathbf{c}^*$ such that $F^*(\theta) = F^e(\theta)$. For some $\theta \in \Theta$, by the definition of optimality for $F^e(\theta) > 0$ we have

$$v_1'(F^e(\theta); \theta_1) + \sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = 1.$$

Thus,

$$\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = 1 - v_1'(F^e(\theta); \theta_1). \tag{A.10}$$

Note that individual 1 has complete information, since she knows the type of all other individuals. Hence, from the first order conditions for this individual, it follows that

$$v_1'(F^e(\theta); \theta_1) = \frac{(c_1(\theta))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{A.11}$$

Thus, equations (A.10) and (A.11) imply that

$$\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = \frac{\sum_{i=2}^{n} (c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}},$$

and so letting $A = \sum_{i=2}^{n} (c_i(\theta_i^1))^{1/2}$ yields the desired result.

To prove the converse, suppose there exists $A \in \mathbb{R}$ for which $\sum_{i=2}^{n} v_i'(F^e(\theta); \theta_i^1) = A(F^e(\theta))^{-1/2}$. Then, let the contribution for an individual $i \in \mathcal{N}$ with type $\theta_i \in \Theta_i$ be given by

$$c_i(\theta_i) = E\left(\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2} \,\middle|\, \theta_i^1 \right] \right)^2, \tag{A.12}$$

which is well-defined since $F^e(\theta)$ exists and is unique. Note that these contributions satisfy the first order conditions if $F(\theta) = F^e(\theta)$, so we must show that this equality. Since every individual $2 \leq i \leq n$ only has a single possible type we have that the conditional expectation in equation (A.12) is equal to the unconditional expectation for all $2 \leq i \leq n$. Taking the square root of both sides of this equation and adding it across all $2 \leq i \leq n$ yields

$$\begin{aligned}\sum_{i=2}^{n}(c_i(\theta_i^1))^{1/2} &= \sum_{i=2}^{n} E\left[v_i'(F^e(\theta)) \cdot (F^e(\theta))^{1/2}\right] \\ &= E\left[(F^e(\theta))^{1/2}\sum_{i=2}^{n} v_i'(F^e(\theta))\right] \\ &= E\left[(F^e(\theta))^{1/2}\frac{A}{(F^e(\theta))^{1/2}}\right] \\ &= A.\end{aligned}$$

Thus, substituting the left-hand side on $\sum_{i=2}^{n} v_i'(F^e(\theta);\theta_i^1) = A(F^e(\theta))^{-1/2}$ we get

$$\sum_{i=2}^{n} v_i'(F^e(\theta);\theta_i^1) = \frac{\sum_{i=2}^{n}(c_i(\theta_i^1))^{1/2}}{(F^e(\theta))^{1/2}} \tag{A.13}$$

On the other hand, since the individual 1 has complete information, rearranging equation (A.12) we get

$$v_1'(F^e(\theta);\theta_1) = \frac{(c_1(\theta_1))^{1/2}}{(F^e(\theta))^{1/2}}. \tag{A.14}$$

Finally, adding equations (A.13) and (A.14) and using the fact that $\sum_{i=1}^{n} v_i'(F^e(\theta);\theta_i) = 1$ by the definition of efficient provision, we get that $F(\theta) = F^e(\theta)$. Thus, it follows that the contributions specified in equation (A.12) do indeed satisfy the first order conditions of an equilibrium, and that this equilibrium is efficient, as we wanted to show.

□

Proof of Proposition 3.12. Suppose that, for all $i \in \mathcal{N}$ and all $\theta \in \Theta$, there exists a function $A_i : \Theta_i \to \mathbb{R}$ such that $v_i'(F^e(\theta);\theta_i) = A_i(\theta_i) \cdot (F^e(\theta))^{-1/2}$. Then, for every $i \in \mathcal{N}$, let the contribution $c_i(\theta_i)$ be defined by

$$c_i(\theta_i) = \left(E\left[v_i'(F^e(\theta);\theta_i) \cdot (F^e(\theta))^{1/2} \,\middle|\, \theta_i\right]\right)^2, \tag{A.15}$$

which is well-defined since $F^e(\theta)$ exists and is unique. These contributions satisfy the first order conditions, and are hence an equilibrium, if $F(\theta) = F^e(\theta)$, which is what we now show. Substituting the hypothesis about A_i in equation (A.15) we get

$$\begin{aligned}(c_i(\theta_i))^{1/2} &= E\left[A_i(\theta_i)\cdot(F^e(\theta))^{-1/2}\cdot(F^e(\theta))^{1/2}\,\middle|\,\theta_i\right]\\ &= A_i(\theta_i),\end{aligned}$$

and thus, again by the hypothesis about A_i, we have

$$v_i'(F^e(\theta);\theta_i) = \frac{(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

So, adding this expression for all i,

$$\sum_{i=1}^{n} v_i'(F^e(\theta);\theta_i) = \frac{\sum_{i=1}^{n}(c_i(\theta_i))^{1/2}}{(F^e(\theta))^{1/2}}.$$

Hence, since $F^e(\theta)$ is efficient, we have that the left-hand side of the above equation is equal to one. We then have that $F(\theta) = \left(\sum_{i=1}^{n}(c_i(\theta_i))^{1/2}\right)^2 = F^e(\theta)$, as we wanted to show. □

Proof of Proposition 3.13. I first show that, if $\gamma = 1/2$, then QF is efficient. If $\gamma = 1/2$, then for all $i \in \mathcal{N}$ we have that $v_i(F;\theta_i) = 2\beta_i(\theta_i)F^{1/2}$ and so $v_i'(F;\theta_i) = \beta_i(\theta_i)F^{-1/2}$. In particular, $v_i'(F^e(\theta);\theta_i) = \beta_i(\theta_i)(F^e(\theta))^{-1/2}$ for all $\theta \in \Theta$, and so the claim follows from proposition 3.12.

To prove the converse, suppose by contradiction that QF is efficient and that $\gamma \neq 1/2$. By hypothesis, we know that there exists some $j \in \mathcal{N}$ and types $\theta_j^k, \theta_j^\ell \in \Theta_j$ such that $\beta_j(\theta_j^k) \neq \beta_j(\theta_j^\ell)$. Without loss of generality, let $\beta_j(\theta_j^k) > \beta_j(\theta_j^\ell)$.

Now, let $\overline{\theta} \in \Theta$ be the state of the world such that $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$, for all $i \in \mathcal{N}$ and all $\theta_i \in \Theta_i$. As we are assuming efficiency, we know that there exists a $\mathbf{c}^*$ which is an equilibrium for the game such that $F^*(\theta) = F^e(\theta)$ for all $\theta \in \Theta$. The first order condition for some individual i in the state of the world $\overline{\theta}$ imply that

$$(c_i^*(\overline{\theta}_i))^{1/2} = \beta_i(\overline{\theta}_i)\cdot E\left[(F^e(\theta))^{1/2-\gamma}\,\middle|\,\overline{\theta}_i\right]. \tag{A.16}$$

We have two cases: $\gamma > 1/2$ or $\gamma < 1/2$. We prove the claim for the first case, and the second case is analogous. First, note that, for any $\theta \in \Theta$, since $\sum_{i=1}^{n} v_i'(F^e(\theta);\theta_i) =$

$\sum_{i=1}^{n} \beta_i(\theta_i) \cdot (F^e(\theta))^{-\gamma} = 1$ by definition of efficiency, then, as $\beta_i(\overline{\theta}_i) \geq \beta_i(\theta_i)$ for all i and all θ_i, it follows that $\sum_{i=1}^{n} v_i'(F^e(\theta); \overline{\theta}_i) = \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\theta))^{-\gamma} \geq 1$, for all $\theta \in \Theta$. As every $v_i'(\cdot; \theta_i)$ is strictly decreasing in F, we have that $F^e(\overline{\theta}) \geq F^e(\theta)$, for all $\theta \in \Theta$. In particular, $F^e(\overline{\theta}) > F^e(\theta_j^\ell, \overline{\theta}_{-j})$. Thus, as $1/2 - \gamma < 0$, it follows that $(F^e(\overline{\theta}))^{1/2-\gamma} \leq (F^e(\theta))^{1/2-\gamma}$ for all $\theta \in \Theta$ and, in particular, $(F^e(\overline{\theta}))^{1/2-\gamma} < (F^e(\theta_j^\ell, \overline{\theta}_{-j}))^{1/2-\gamma}$. Since $\Pr\left(\theta_j^\ell \,\middle|\, \overline{\theta}_i\right) > 0$ for all $i \neq j$, we then have

$$E\left[F^e(\theta))^{1/2-\gamma} \,\middle|\, \overline{\theta}_i\right] \geq (F^e(\overline{\theta}))^{1/2-\gamma} \tag{A.17}$$

for all $i \in \mathcal{N}$, and the inequality is strict when $i \neq j$. By (A.16) and (A.17) we have that

$$(c_i^*(\overline{\theta}_i))^{1/2} \geq \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\gamma} \tag{A.18}$$

for all $i \in \mathcal{N}$, with strict inequality when $i \neq j$. Adding (A.18) across all $i \in \mathcal{N}$ yields

$$\sum_{i=1}^{n} (c_i^*(\overline{\theta}_i))^{1/2} = (F^e(\overline{\theta}))^{1/2} > \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{1/2-\gamma}. \tag{A.19}$$

Dividing both sides of the inequality in (A.19) by $(F^e(\overline{\theta}))^{1/2}$, we get

$$1 > \sum_{i=1}^{n} \beta_i(\overline{\theta}_i) \cdot (F^e(\overline{\theta}))^{-\gamma} = \sum_{i=1}^{n} v_i'(F^e(\overline{\theta}); \overline{\theta}_i), \tag{A.20}$$

which is a contradiction to the definition of efficient provision. This contradiction completes the proof. □

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